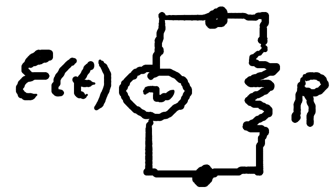
Recap

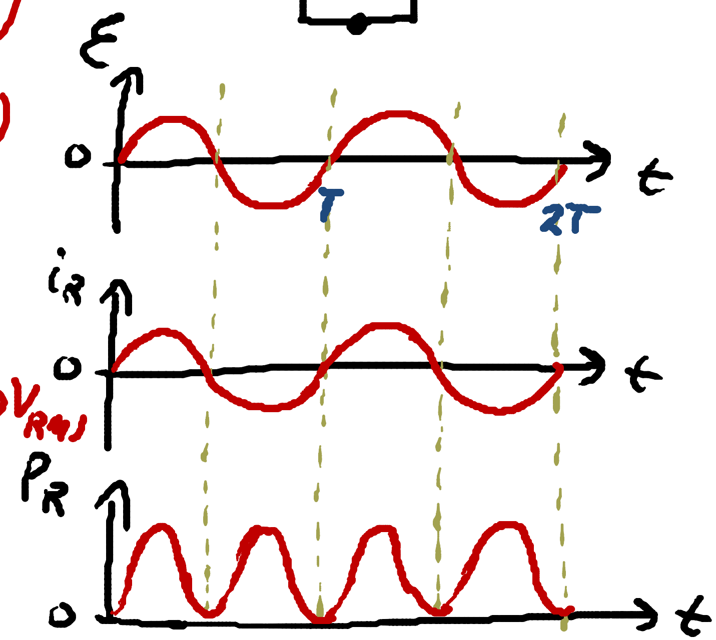
Lecture 23

- Magnetic field energy:
  - Energy stored in an inductor: $U_B = \frac{1}{2} L i^2$
  - Energy density of a magnetic field: $u_B = \frac{1}{2}\frac{B^2}{\mu_0}$
- Alternating current (ac):
  - $\mathcal{E}(t) = \mathcal{E}_{max} \sin(2\pi f t)$
  - $i_R(t) = \underbrace{i_{max}}_{= \mathcal{E}_{max}/R} \sin(2\pi f t)$
  - average power dissipated in resistor:

    $$P_{avg} = \left(\frac{i_{max}}{\sqrt{2}}\right)^2 R = i_{RMS}^2 R = i_{RMS} \Delta V_{RMS}$$
  - RMS values:

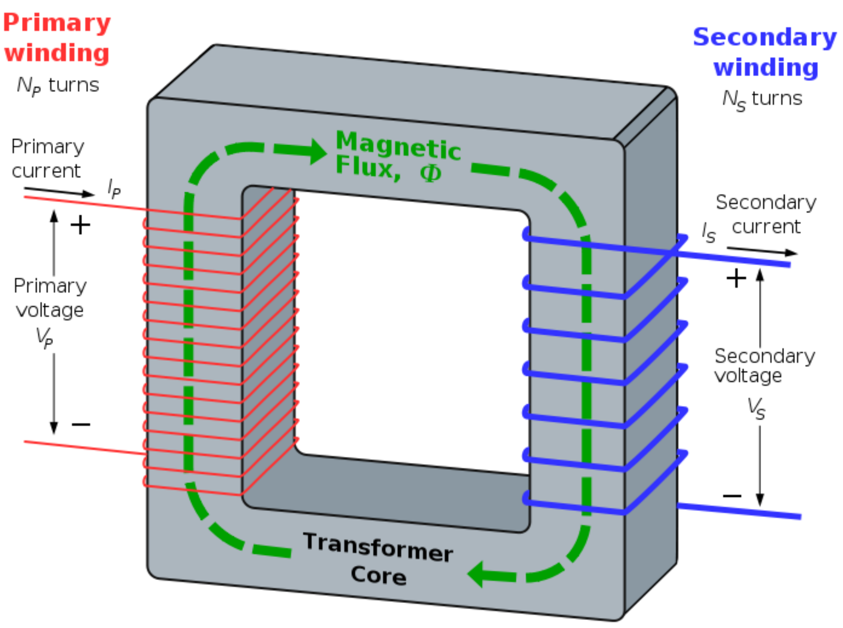
    $$i_{RMS} = \frac{i_{max}}{\sqrt{2}} \qquad \Delta V_{RMS} = \frac{\Delta V_{max}}{\sqrt{2}}$$

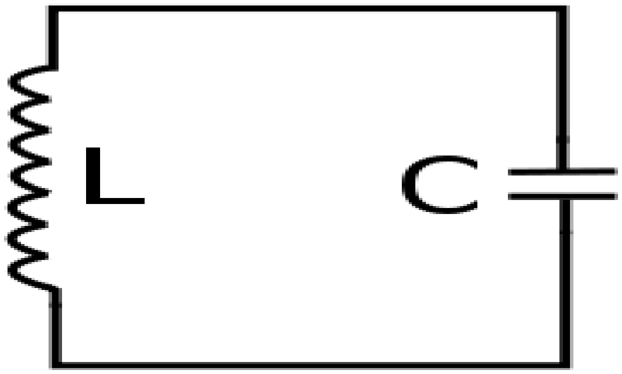




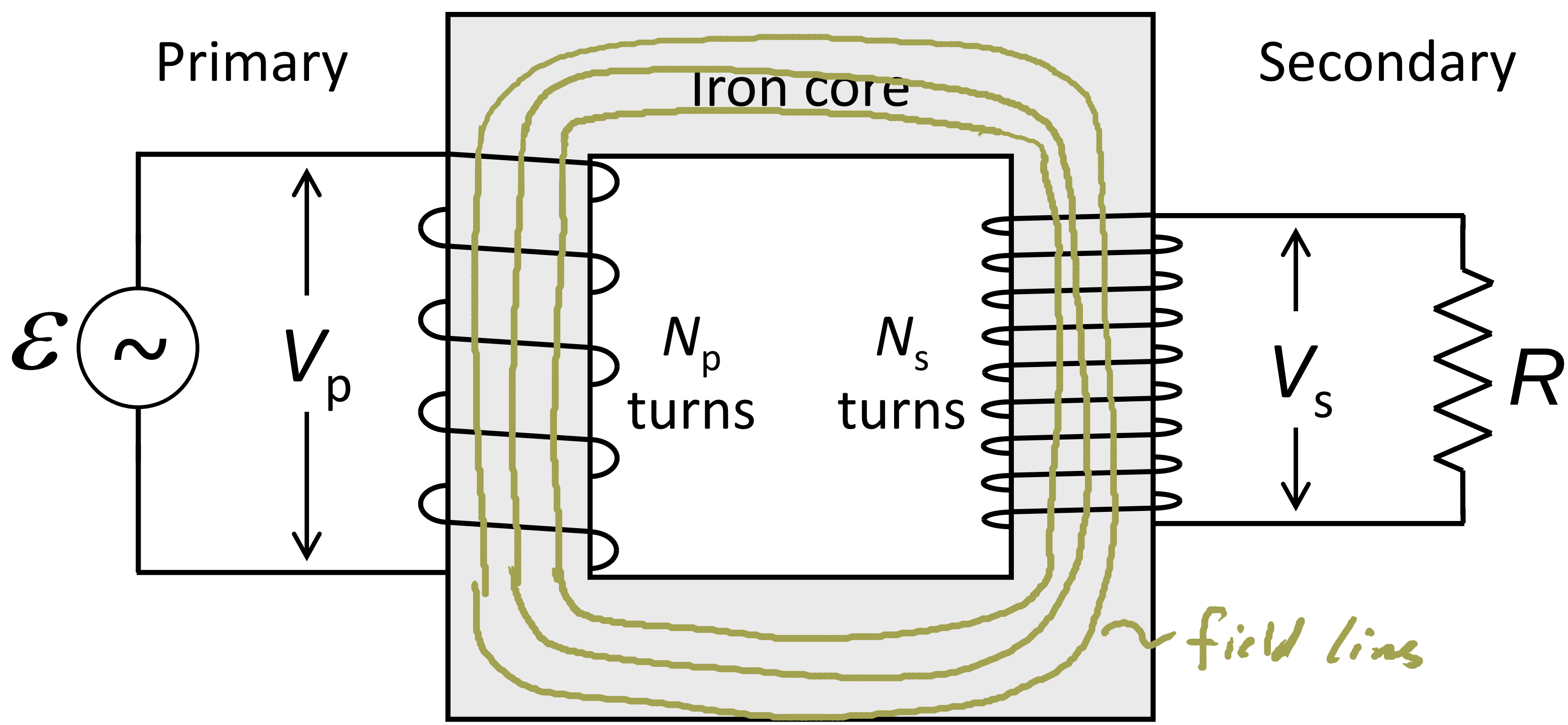
Today:

- Alternating current and power
  - Transformers
- Ideal LC circuit
- RLC circuit: damping and driven





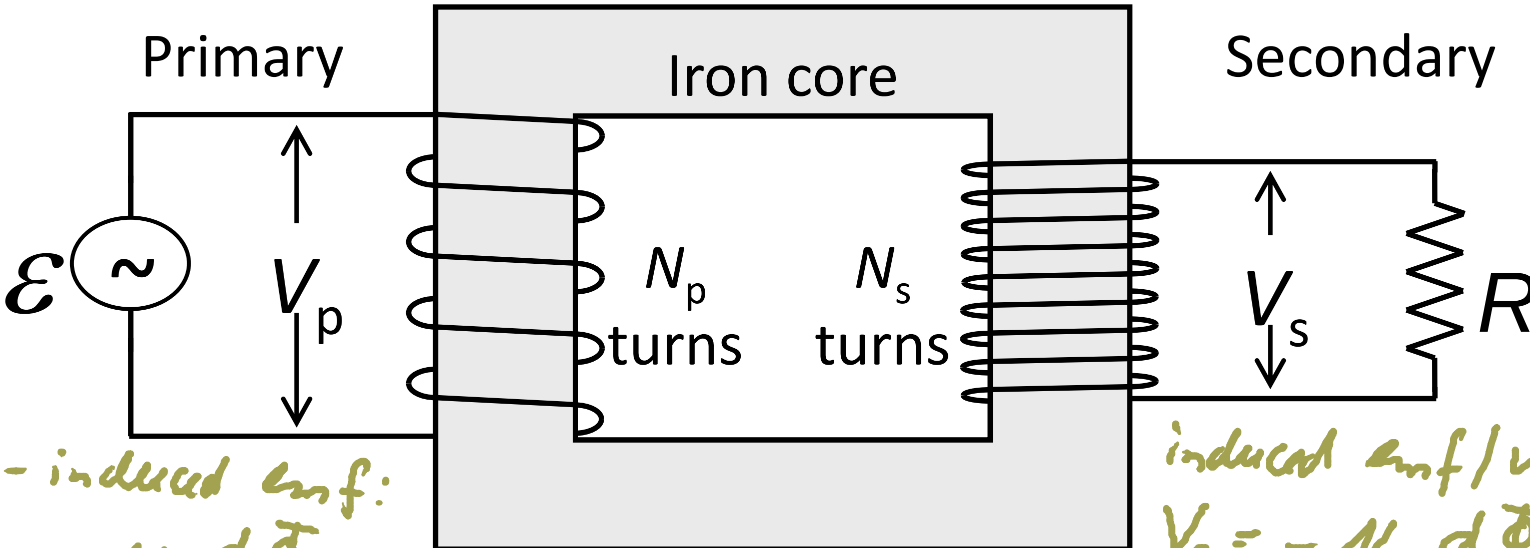
Transformer:



The iron core ensures that the **Φ_B per turn is the same in both the primary & secondary windings**.

$$\Phi_{B,\text{primary}} = \Phi_{B,\text{secondary}} \Rightarrow \frac{d\Phi_{B,\text{pr}}}{dt} = \frac{d\Phi_{B,\text{sec}}}{dt}$$

Transformer:



Self-induced emf:

$$V_p = -N_p \frac{d\Phi_B}{dt}$$

induced emf/voltage:

$$V_s = -N_s \frac{d\Phi_B}{dt}$$

$$\Rightarrow \frac{V_s}{V_p} = \frac{N_s}{N_p} \underbrace{\frac{d\Phi_B/dt}{d\Phi_B/dt}}_{=1} = \frac{N_s}{N_p}$$

ratio of turns determines output voltage V_s

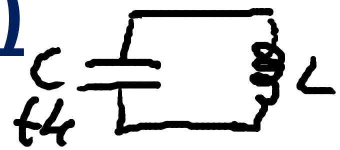
- if $N_s > N_p$: step-up transformer
- if $N_s < N_p$: step-down transformer

$\Rightarrow$ Conservation of energy requires:

$$P_{in} = i_p V_p = P_{out} = i_s V_s$$

$$\Rightarrow i_s = i_p \frac{V_p}{V_s} = i_p \frac{N_p}{N_s}$$

Ideal LC – circuit (no resistance)



energy stored in the electric field of capacitor

$$U_E = \frac{1}{2}\frac{q^2}{C}$$

energy stored in the magnetic field of the inductor

$$U_B = \frac{1}{2} L i^2$$

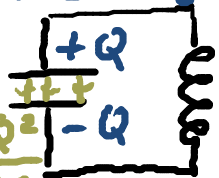
⇒ total energy stored:

$$\boxed{U_{total} = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2} L i^2 = \text{const}}$$

- energy is conserved, so this total sum is constant, if there is no resistance in the circuit!
- But: energy can oscillate back and forth between electric field energy in the capacitor and magnetic field energy in the inductor!

Electromagnetic Oscillations in ideal LC circuit:

capacitor starts with charge $q = Q$



$U_B = 0$

$U_E = \frac{Q^2}{2C}$

$i = 0$

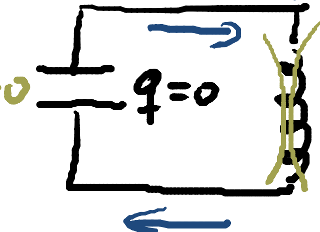
all energy is in the electric field

current starts to flow and C discharges

L regulates the increase in current

current increases until $q = 0$

$i = i_{max}$



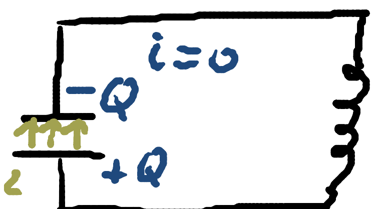
$U_E = 0$

$U_B = \frac{1}{2} L i_{max}^2$

all energy is in the magnetic field

L tries to keep the current going, so C gets charged again, but with opposite polarity

current decreases to $i = 0$



$U_B = 0$

$U_E = \frac{Q^2}{2C}$

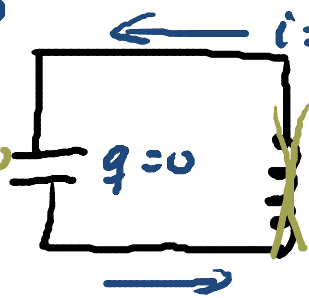
all energy is in the electric field

current starts to flow and C discharges

L regulates the increase in current

current increases until $q = 0$

$i = i_{max}$



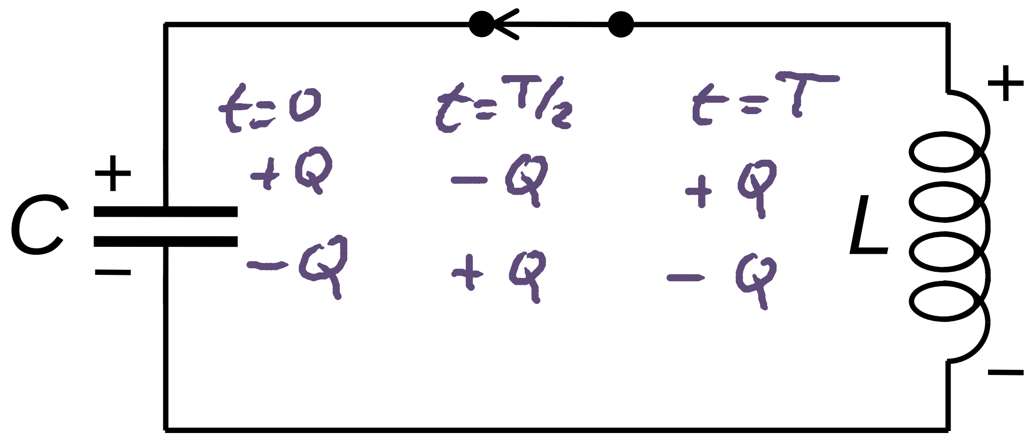
$U_E = 0$

$U_B = \frac{1}{2} L i_{max}^2$

all energy is in the magnetic field

L tries to keep the current going, so C charges up again

circuit oscillates!

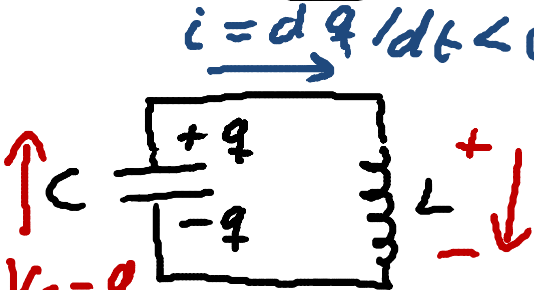


The capacitor starts with charge Q. At time $t = 0$ the switch is closed. Let T represent the period of the circuit oscillations.

What is the charge on the capacitor at time $T/2$?

A. 0 B. $+Q/2$ C. $-Q/2$ D. $+Q$ E. $-Q$

Ideal LC-circuit (no resistance):



$\Delta V_L = L \dfrac{di}{dt} = L \dfrac{d}{dt}\dfrac{dq}{dt} = L \underbrace{\dfrac{d^2 q}{dt^2}}_{\text{2nd derivative wrt time}} < 0$

q = charge on capacitor

since $i = \dfrac{dq}{dt}$

$\Delta V_C = \dfrac{q}{C} > 0$

$\Rightarrow$ loop rule gives: $\Delta V_C + \Delta V_L = 0$

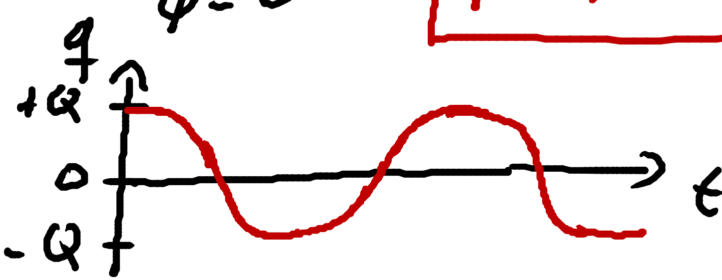
$$\Rightarrow \quad \left.\frac{q}{C} + L\frac{d^2 q}{dt^2} = 0 \quad \right\}$$ differential equation for oscillation of charge q in LC circuit

Solution:

$\boxed{q(t) = Q \cos(\omega t + \phi)}$ } same oscillation as in SHM!

max. charge on capacitor during oscillation

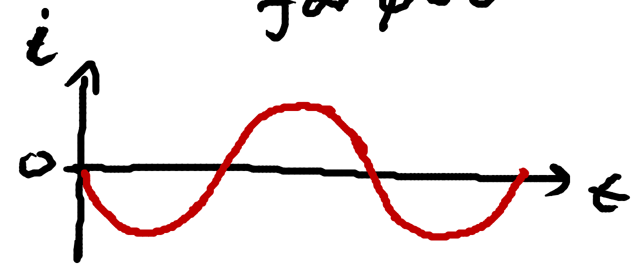
$\phi = 0$



⇒ <u>current</u> in LC circuit:

$$\boxed{i = \frac{dq}{dt} = -\omega Q \sin(\omega t + \phi)}$$

for $\phi = 0$



⇒ <u>find angular frequency</u> ω of oscillation:

need: $\dfrac{d^2q}{dt^2} = \dfrac{di}{dt} = -\omega^2 Q \cos(\omega t + \phi)$

insert into differential equation: $\dfrac{q}{C} + L\dfrac{d^2q}{dt^2} = 0$

$$\frac{Q}{C}\cos(\omega t + \phi) - L\omega^2 Q \cos(\omega t + \phi) = 0$$

gives: $\dfrac{1}{C} - L\omega^2 = 0 \Rightarrow \boxed{\omega_{0,LC} = \dfrac{1}{\sqrt{LC}}}$

natural angular frequency for LC-circuit without damping

$\Rightarrow$ total energy in LC oscillator:

$$U_{total} = U_E + U_B = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}L\,i^2 \qquad \omega^2 = \frac{1}{LC}$$

$$= \frac{1}{2}\frac{Q^2}{C}\cos^2(\omega t + \phi) + \frac{1}{2}L\,\omega^2 Q^2 \sin^2(\omega t + \phi)$$

$$= \frac{1}{2}\frac{Q^2}{C}\Big(\underbrace{\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)}_{=1}\Big)$$

$$= \frac{1}{2}\frac{Q^2}{C} = U_{E,\,max} = U_{B,\,max} = \underline{\underline{\text{const}}}$$

$\Rightarrow$ Energy oscillates between electric and magnetic fields, but the total sum remains constant!

(like for SHM of a mass on a spring: energy oscillates between kinetic and potential energy)