Lecture 4

2. Charged particles in magnetic fields

2.1 Basics

2.2 Magnets

2.3 Multipole expansion

2.4 Superconducting magnets

2.1 Basics

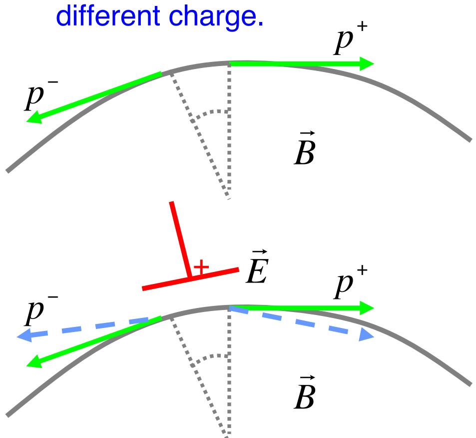
Lorentz force
Maxwell's equations
Magnetic boundary conditions

Macroscopic Fields in Accelerators

$$\frac{d}{dt}\vec{p} = q(\vec{E} + \vec{v} \times \vec{B})$$

E has a similar effect as v B.
For relativistic particles B = 1T has a similar effect as E = cB = 3 10^8 V/m , such an electric field is beyond technical limits.

- Electric fields are only used for very low energies or
- For separating two counter rotating beams with different charge.




Electrostatic separators at CESR

Maxwell's equations (I)

in integral form:

$$\oint_A \vec{E} \cdot d\vec{a} = \int_V \frac{\rho}{\varepsilon_r \varepsilon_0} dV \quad \Big\} \text{ Gauss's Law}$$

charge density

$$\oint_A \vec{B} \cdot d\vec{a} = 0 \quad \Big\} \text{ Gauss's Law for magnetic field}$$

$$\oint \vec{E} \cdot d\vec{s} = -\int_A \dot{\vec{B}} \cdot d\vec{a} = -\frac{\partial \Phi}{\partial t} \quad \Big\} \text{ Law of induction}$$

$$\oint \vec{B} \cdot d\vec{s} = \int_A \mu_r \mu_0 (\vec{j} + \varepsilon_r \varepsilon_0 \dot{\vec{E}}) \cdot d\vec{a} \quad \Big\} \text{ Ampère - Maxwell Law}$$

Maxwell's equations (II)

in differential form:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_r\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_r\mu_0\left(\vec{j} + \varepsilon_r\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

electrical displacement $\vec{D} = \varepsilon_r\,\varepsilon_r\,\vec{E}$
↖ permittivity

magnetizing field: $\vec{H} = \frac{1}{\mu_r\mu_0}\vec{B}$
↖ permeability

Static magnetic fields in accelerators

static: $\frac{\partial\vec{B}}{\partial t} = 0$, $\vec{E} = 0$

charge free space near beam: $\vec{j} = 0$, $\mu_r = 1$, $\varepsilon_r = 1$

$$\vec{\nabla}\times\vec{B} = \mu_r\mu_0\left(\vec{j} + \varepsilon_r\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) = 0$$

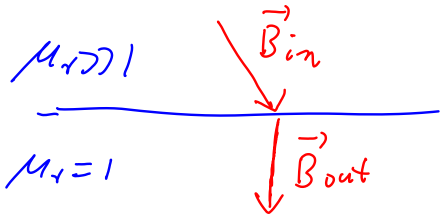
$\Rightarrow$ $\vec{B}$ can be written as the gradient of a scalar potential: $\psi(\vec{r})$: $\boxed{\vec{B} = -\vec{\nabla}\psi(\vec{r})}$

(since $\vec{\nabla}\times\vec{\nabla}\psi = 0$ always)

$\Rightarrow$ also: $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \boxed{\vec{\nabla}^2\psi(r) = 0}$

(x=0,z=0) is the beam's design curve

Boundary between vacuum and a material with high permeability (I)



$$B_{\perp,out} = B_{\perp,in} \Leftrightarrow H_{\perp,out} = \mu_r H_{\perp,in}$$

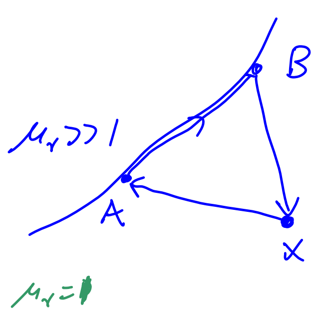
$$B_{\parallel,out} = \frac{1}{\mu_r} B_{\parallel,in} \Leftrightarrow H_{\parallel,out} = H_{\parallel,in}$$

$\Rightarrow$ for large permeability (e.g. iron with $\mu_r > 1000$) $\vec{H}_{out}$ and $\vec{B}_{out}$ are $\perp$ to the surface!

$\Rightarrow$ use shape of magnetic pole to define magnetic field shape near beam axis!

Boundary between vacuum and a material with high permeability (II)

$$\vec{\nabla} \times \vec{B} = 0 \Rightarrow \vec{B}(\vec{r}) = -\vec{\nabla}\Psi(\vec{r})$$



$\Rightarrow$ for closed path: total potential difference traversed = 0

$$\underline{0} = \oint \vec{B} \cdot d\vec{s}$$

$$= \int_X^A \vec{B} \cdot d\vec{s} + \underbrace{\int_A^B \vec{B} \cdot d\vec{s}}_{\approx 0,\text{ since } \vec{B} \perp \text{ at surface}} + \int_B^X \vec{B} \cdot d\vec{s}$$

$$\approx \int_X^A \vec{B} \cdot d\vec{s} + \int_B^X \vec{B} \cdot d\vec{s} = \underline{\Psi(A) - \Psi(B)}$$

$\Rightarrow$ Surfaces of materials with high permeability form an equipotential ($\Psi = \text{const}$)

Green's Theorem

$$\vec{\nabla}^2\psi = 0$$

Green function: $\vec{\nabla}_0^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$

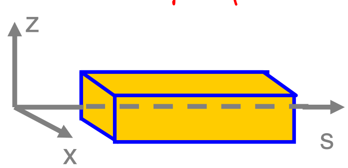
$$\psi(\vec{r}) = \int_V \psi(\vec{r}_0)\delta(\vec{r}-\vec{r}_0)\,d^3\vec{r}_0$$
$$= \int_V \left[\psi(\vec{r}_0)\vec{\nabla}_0^2 G - G\vec{\nabla}_0^2\psi(\vec{r}_0)\right]d^3\vec{r}_0$$
$$= \int_V \vec{\nabla}_0\left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right]d^3\vec{r}_0$$
$$= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right]\cdot d^2\vec{r}_0$$
$$= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G + \vec{B}(\vec{r}_0)G\right]\cdot d^2\vec{r}_0$$

Knowledge of the field and the scalar magnetic potential on a closed surface inside a magnet determines the magnetic field for the complete volume which is enclosed. -> shape of magnetic poles fixes magnetic field pattern near beam line!

Potential Expansion

If field data in a plane (for example the horizontal midplane of a cyclotron or of a beam line magnet) is known, the complete ~~filed~~ field is determined:

$$\psi(x,z,s) = \sum_{n=0}^{\infty} b_n(x,s)z^n \quad \Rightarrow \quad \vec{B}(x,0,s) = -\begin{pmatrix} \partial_x b_0(x,s) \\ b_1(x,s) \\ \partial_s b_0(x,s) \end{pmatrix}$$



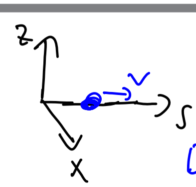
$$0 = \vec{\nabla}^2\psi = \sum_{n=0}^{\infty}(\partial_x^2 + \partial_s^2)\,b_n\,z^n + \sum_{n=2}^{\infty} n(n-1)\,b_n\,z^{n-2}$$
$$= \sum_{n=0}^{\infty}\left[(\partial_x^2+\partial_s^2)\,b_n + (n+2)(n+1)\,b_{n+2}\right]z^n$$
$$b_{n+2}(x,s) = -\frac{1}{(n+2)(n+1)}(\partial_x^2+\partial_s^2)b_n(x,s)$$

Data of the magnetic field in the plane z=0 is used to determine $b_0(x,s)$ and $b_1(x,s)$.

2.2 Magnets

solenoid
dipole
quadrupole
sextupole
combined function

Multipole expansion of $B_z(x,z=0)$



expand magnetic field in vicinity of beam trajectory:

$$B_z(x, z=0) = B_{z0} + \left.\frac{dB_z}{dx}\right|_0 x + \frac{1}{2!}\left.\frac{d^2B_z}{\partial x^2}\right|_0 x^2 + \frac{1}{3!}\left.\frac{d^3B_z}{\partial x^3}\right|_0 x^3 + \ldots$$

$$\Rightarrow \frac{q}{p}B_z(x, z=0) = \frac{q}{p}B_{z0} + \frac{q}{p}\left.\frac{dB_z}{dx}\right|_0 x + \frac{q}{p}\frac{1}{2!}\left.\frac{d^2B_z}{dx^2}\right|_0 x^2 + \frac{q}{p}\frac{1}{3!}\left.\frac{d^3B_z}{dx^3}\right|_0 x^3 + \ldots$$

$$= \frac{1}{R} + kx + \frac{1}{2!}m\,x^2 + \frac{1}{3!}o\,x^3 + \ldots$$

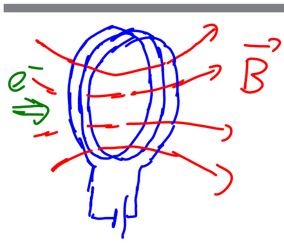
= dipole + quadrupole + sextupole + octupole + ...

Magnetic Multipoles in Accelerators

multipole	strength	effect
dipole	$\frac{1}{R} = \frac{q}{p} B_{z0}$	beam steering
quadrupole	$k = \frac{q}{p} \frac{dB_z}{dx}$	beam focusing
sextupole	$m = \frac{q}{p} \frac{d^2 B_z}{dx^2}$	chromaticity compensation
octupole	$o = \frac{q}{p} \frac{d^3 B_z}{dx^3}$	field errors, field error compensation

$\Rightarrow$ linear beam optics: dipole + quadrupoles only

Solenoids {sin(0·φ)-dependence}



$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow$ magnetic field must contain radial components in outer part off axis

$\Rightarrow$ particles entering off axis experience azimuthal acceleration while entering and leaving lens

$\Rightarrow$ azimuthal motion (rotation)

$\Rightarrow$ radial force in the longitudinal solenoid field $\propto$ offset r from axis (short lens)

$\Rightarrow$ focusing

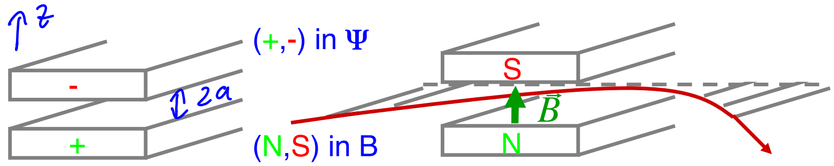
focal length: $\frac{1}{f_{sol}} = \int \left(\frac{q B_s}{2p} \right)^2 ds$

Note: $f_{sol} \propto p^2 \Rightarrow$ effective for small momenta only

Dipoles {sin(1·φ)-dependence}

C$_1$ Symmetry

(C_n Symmetry around the ξ-axis: sign change after $\Delta\varphi = \frac{\pi}{n}$)



homogeneous field: $\vec{B} = B_0 \, \vec{e}_z$

$\Rightarrow$ required potential: $\underline{\psi = -B_0 z} \Leftrightarrow \vec{B} = -\vec{\nabla}\psi = B_0 \, \vec{e}_z$

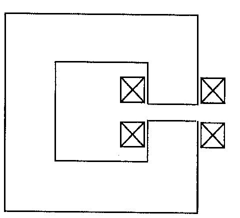
$\Rightarrow$ Equipotentials: $z = \text{const}$

$\Rightarrow$ two horizontal iron poles, spaced by $2a$

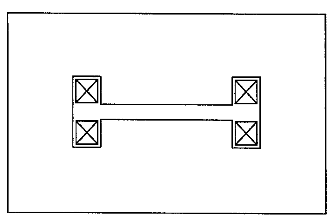
$\Rightarrow$ bending radius: $R = \frac{p}{q B_0}$

Different Dipole Magnets

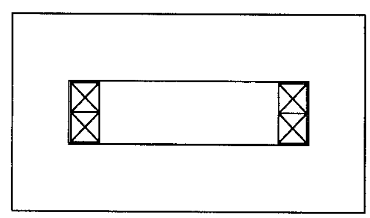
C-shape magnet:

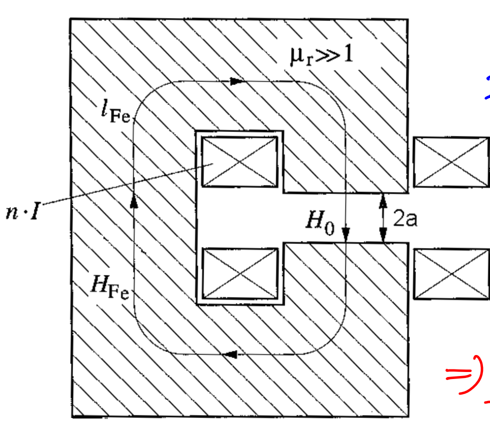


H-shape magnet:



Window frame magnet:





at surface of pole: $H_{\perp,0} = \mu_r H_{\perp,Fe}$

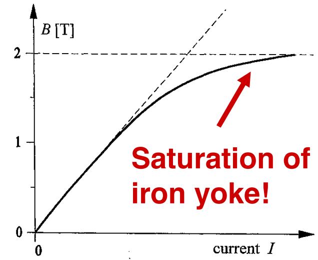
$2nI = \oint \vec{H} \cdot d\vec{s} = H_0 2a + H_{Fe} \ell_{Fe}$

$= H_0 2a + \frac{1}{\mu_r} H_0 \ell_{Fe} \underset{\mu_r \gg 1}{\approx} H_0 2a$

$\Rightarrow \boxed{B_0 = \mu_0 \frac{nI}{a}}$ (neglecting fringe fields, iron saturation)

$\Rightarrow$ dipole strength $\frac{1}{R} = \frac{q}{p} B_0 = \frac{q}{p} \mu_0 \frac{nI}{a}$

Dipole Fields: Limitations

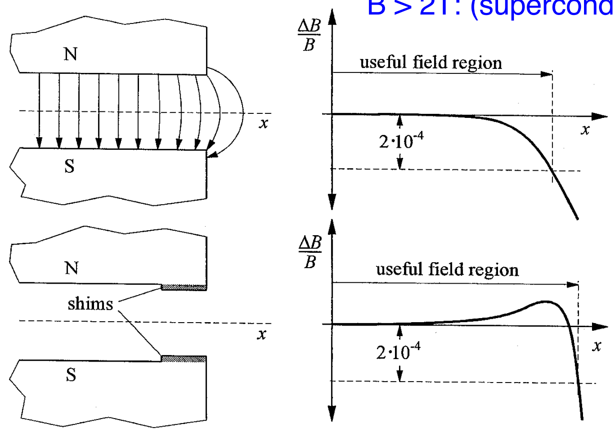


B < 1 T: Region in which $B_0 = \mu_0 \dfrac{nI}{a}$

B < 1.5 T: Typically used region

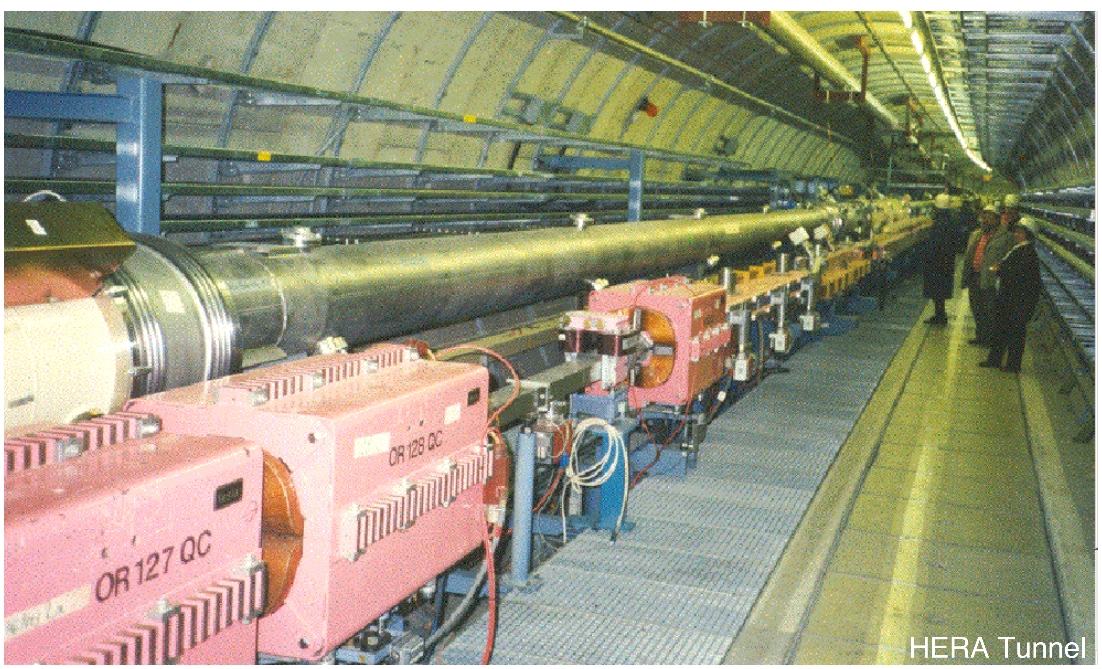
B = 2 T: Typical limit of iron yoke magnets, since the field becomes dominated by the coils, not the iron. Limiting j for Cu is about 100A/mm²

B > 2T: (superconducting) "air coil" magnets



Shims reduce the space that is open to the beam, but they also reduce the fringe field region.

Where is the vertical Dipole?



Quadrupoles {sin(2·φ)-dependence}

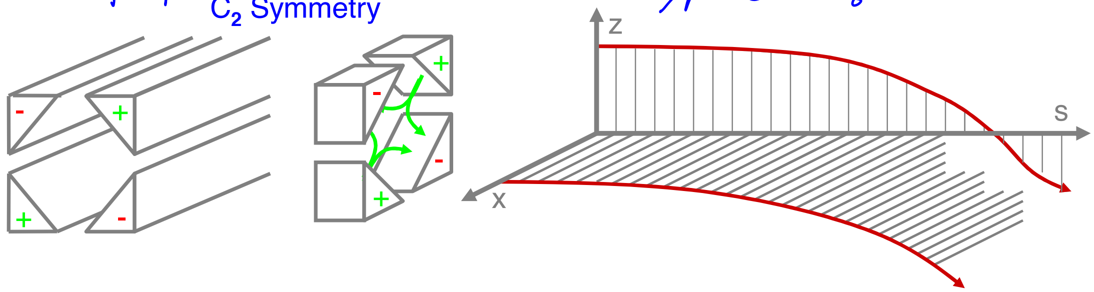
to focus beam: fields disappear along beam axis and increase linear in the deviation from the axis

$$\underbrace{B_z = -g\,x}_{\text{focusing } (g>0)} \qquad \underbrace{B_x = -g\,z}_{\text{defocusing}}$$

$\Rightarrow$ required potential: $\psi(x,z) = g\,x\,z \Leftrightarrow \vec{B} = -\vec{\nabla}\psi = -g\begin{pmatrix} z \\ x \end{pmatrix}$

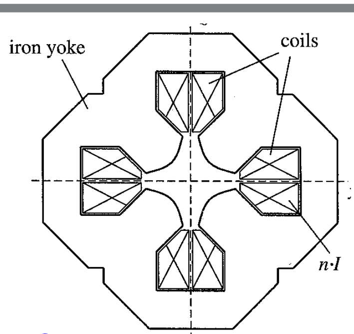
$\Rightarrow$ equipotentials: $x\,z = \text{const}$: hyperbolic !

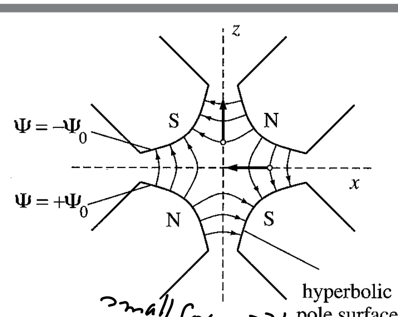
C_2 Symmetry



In a quadrupole particles are focused in one plane and defocused in the other plane. Other modes of strong focusing are not possible.

Quadrupole Fields





equipotentials:
$x \cdot z = \text{const}$

$\Rightarrow$ 4 iron poles with hyperbolic surfaces (N-S-N-S)

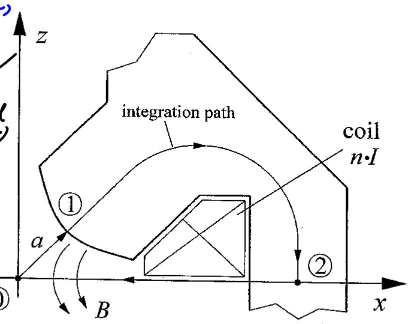
field:

$$nI = \oint \vec{H}\cdot d\vec{s} = \int_0^a H(r)\,dr + \underbrace{\int_1^2 \vec{H}_{FE}\cdot d\vec{s}}_{\text{small for } \mu_r \gg 1} + \underbrace{\int_2^0 \vec{H}\cdot d\vec{s}}_{0 \text{ since } \vec{H} \perp d\vec{s}}$$

$$|H(r)| = \frac{g}{\mu_0}\sqrt{x^2+y^2} = \frac{g}{\mu_0}\,r$$

$$\Rightarrow nI \approx \int_0^a g/\mu_0\, r\,dr \Rightarrow g = \frac{2\mu_0 nI}{a^2}$$

$$\Rightarrow \text{Quadrupole strength: } k = \frac{q}{p}\,g = \frac{q}{p}\,\frac{2\mu_0 nI}{a^2}$$



Quadrupole Fields (II)

$\Rightarrow$ quadrupole of length ℓ:

focal length: $\frac{1}{f} = k\ell$

thin lens: $f \gg \ell$

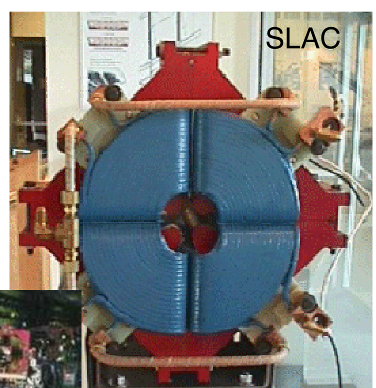
Note: in quadrupole:

$$F_x = q v B_z = -q v g x = f(x)$$

$$F_z = -q v B_x = q v g z = f(z)$$

$\Rightarrow$ in linear beam optics (dipoles + normal (unrotated) quadrupoles): horizontal and vertical motion are decoupled!

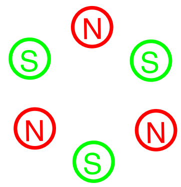
Real Quadrupoles



The coils show that this is an upright quadrupole not a rotated or skew quadrupole.

Sextupoles {sin(3·φ)-dependence}

C_3 Symmetry



⇒ non-linear field, couple x- and z-motion

⇒ used primarily to correct "chromatic errors" resulting from particle momentum dependent focal length of quadrupoles

$$\vec{B} = g' \begin{pmatrix} xz \\ \frac{1}{2}(x^2 - z^2) \end{pmatrix}$$

⇒ $x \to x + \Delta x$

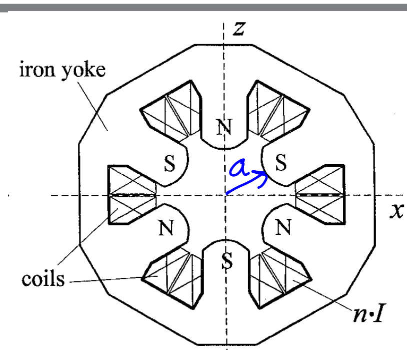
$$\vec{B} \approx g' \begin{pmatrix} xz \\ \frac{1}{2}(x^2 - z^2) \end{pmatrix} + g' \Delta x \begin{pmatrix} x \\ z \end{pmatrix}$$

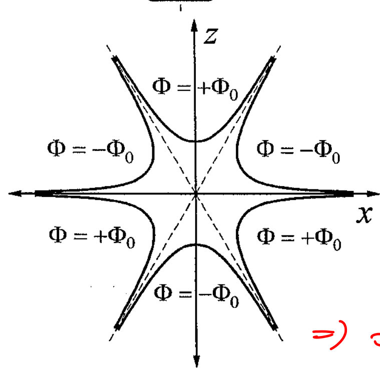
i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

Sextupole fields





⇒ required potential:

$$\psi(x, z) = -\frac{1}{2} g' \left(x^2 z - \frac{1}{3} z^3 \right)$$

⇒ equipotentials:

$$x = \sqrt{\frac{\text{const}}{z} + \frac{1}{3} z^2}$$

⇒ iron with 6 poles, each at 60° angle to next with alternating polarity

⇒ field:

$$nI = \oint \vec{H} \cdot d\vec{s} \approx \frac{g'}{6} \frac{a^3}{\mu_0}$$

⇒ $g' = 6 \mu_0 \frac{nI}{a^3}$

⇒ sextupole strength: $m = \frac{q}{p} g' = \frac{q}{p} 6 \mu_0 \frac{nI}{a^3}$

Real Sextupole

Sextupole in the CESR tunnel