Lecture 24

6. Synchrotron radiation and radiative damping effects

6.1 Synchrotron radiation (in bends)

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→ radiation of e^- in bending magnet → accel → emitts radiation

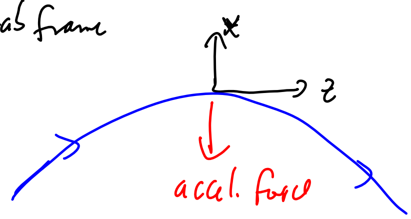
a) In co-moving frame ⇒ $v \ll c$

• total radiated power by charge e: (Larmor equ.; see Jackson)

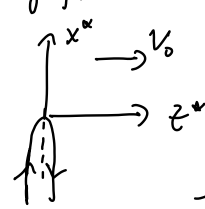
$$P_\gamma^* = \frac{e^2}{6\pi \varepsilon_0 c^3}\left(\frac{d\vec{v}^*}{dt^*}\right)^2 \Rightarrow |P_\gamma^*| > 0 \text{ if } \vec{a}^* \neq 0$$

• angular distribution: ≈ Hertz dipole

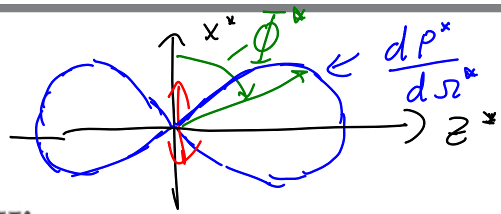
Lab frame

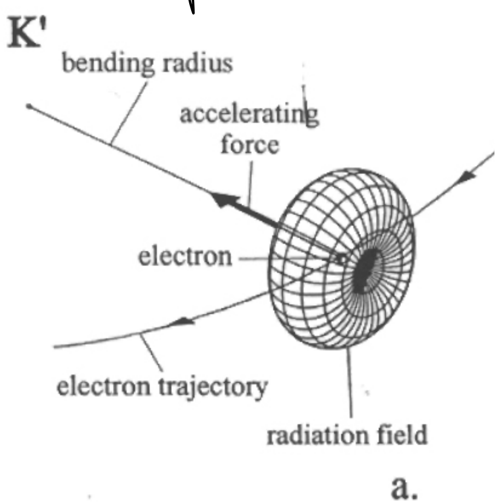


co-moving frame



like e^- in Hertz dipole $\sin^2\Phi$ dep.



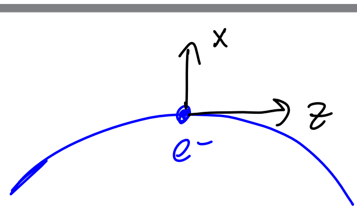


$$\frac{dP^*}{d\Omega^*} = \frac{e^2}{16\pi^2\varepsilon_0 c^3}\left(\frac{d\vec{v}^*}{dt^*}\right)^2 \sin^2\Phi^*$$

with

$$d\Omega^* = \sin\Phi^* \, d\Phi^* \, d\theta^*$$

b) Radiation from relativistic particle



E^* = energy of e^- in co-moving frame

E = energy of e^- in Lab frame

$\rightsquigarrow E = E^* \gamma \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$

$dt = dt^* \gamma$: time dilation

$\rightsquigarrow$ Synchrotron radiation power emitted:

$$P_\gamma = -\frac{dE}{dt} = -\frac{dE^*}{dt^*} = P_\gamma^*$$

$\Rightarrow$ radiated power is relativistically invariant!

$$\Rightarrow P_\gamma = \frac{e^2}{6\pi\varepsilon_0 c^3}\left(\frac{d\vec{v}^*}{dt^*}\right)^2$$

- Lorentz transformation of acceleration $\vec{a}^* = \dot{\vec{v}}^*$

$$\underbrace{\dot{v}_\parallel^* = \gamma^3 \dot{v}_\parallel}_{\text{component} \parallel \text{ to } \vec{v}} \qquad \underbrace{\dot{v}_\perp^* = \gamma^2 \dot{v}_\perp}_{\text{component} \perp \text{ to } \vec{v}}$$

(from $z^* = \gamma(z - vt)$, $x^* = x$, $y^* = y$, $t^* = \gamma(t - \frac{v}{c^2}x)$)

$\Rightarrow$ for accelerators, consider two cases:

1.) linear acceleration $\quad \frac{d\vec{v}}{dt} \parallel \vec{v}$

2.) Circular acceleration (bending magnet) $\quad \frac{d\vec{v}}{dt} \perp \vec{v}$, $|v| = \text{const}$

c) Liner acceleration

$$\frac{dv_\parallel^*}{dt^*} = \gamma^3 \frac{dv_\parallel}{dt} = \gamma^3 a_\parallel = \frac{dp}{dt}\frac{1}{m_0} = \frac{1}{m_0}\frac{dE}{dx}$$

$$\Rightarrow P_\gamma = \frac{e^2}{6\pi\varepsilon_0 m_0^2 c^3}\left(\frac{dE}{dx}\right)^2$$

Example: e^-, $\frac{dE}{dx} = 15\frac{MeV}{m} \Rightarrow P_\gamma \approx 4 \cdot 10^{-17}\,W \Rightarrow$ tiny!

$\Rightarrow$ radiation of electromagnetic power during longitudinal accel. is negligible small!

$\Rightarrow$ relative loss: $\eta = \frac{P_\gamma}{(dE/dt)_{accel}} = \frac{P_\gamma}{v\frac{dE}{dx}} = \frac{e^2}{6\pi\varepsilon_0 m_0^2 c^4 \beta}\frac{dE}{dx}$

$\Rightarrow$ for example: $\eta = 5.5 \cdot 10^{-14}$ for $\beta \approx 1$

d) Circular acceleration

accel. due to magnetic bending field: $|\vec{v}| = \text{const}$

$$\frac{dv_\perp^*}{dt^*} = \gamma^2 \frac{dv_\perp}{dt} = \gamma^2 a_\perp = \gamma^2 \frac{v^2}{\rho}$$

ρ: bending radius of particle orbit

=) radiated power during transverse acceleration

$$P_s = \frac{e^2 c}{6\pi \varepsilon_0} \frac{\beta^4 \gamma^4}{\rho^2} \propto \underline{\underline{\gamma^4}}$$

(first observed in synchrotrons → "synchrotron radiation")

=) since $\gamma = E/m_0c^2$

$$P_s = \frac{e^2 c}{6\pi \varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho^2}$$

Note:

1) During energy ramp-up in synchrotron ($\rho = \text{const}$)

$$P_s \propto E^4$$

2) $E = pc$ (highly relativistic)

$$= e B_{dipole} \rho c$$

$$\Rightarrow P_s = \frac{e^2 c}{6\pi \varepsilon_0 (m_0 c^2)^4} E^2 (e c B_{dipole})^2 \propto E^2 B_{dipole}^2$$

$\Rightarrow$ for given B_{dipole}: $P_s \propto E^2$
(ρ changes here!)
(case for particles with $\delta \neq 0$ in given accelerator)

3) compare power radiated by e^- ($m_e c^2 = 511\,keV$) with power radiated by proton ($m_p c^2 = 938\,MeV$) at same energy:

$$\frac{P_{s,e}}{P_{s,p}} = \left(\frac{m_p c^2}{m_e c^2}\right)^4 = \underline{\underline{10^{13}}}$$

$\Rightarrow$ synchrotron radiation only relevant for electrons!

4) total energy loss to radiation during one complete turn:

$$\Delta E_s = \oint P_s\, dt = \frac{1}{c} \underbrace{\oint P_s\, ds}_{\text{bending magnets}} = P_s \frac{2\pi\rho}{c}$$

$$\Rightarrow \Delta E_s = \frac{e^2}{3\varepsilon_0 (m_0 c^2)^4} \frac{E^4}{\rho}$$

$\Rightarrow$ for electrons: $\frac{\Delta E}{keV} = 88.5 \frac{(E/GeV)^4}{\rho/m}$

$\Rightarrow$ $\Delta E_s \propto E^4$ $\quad \Delta E \propto \frac{1}{\rho}$

⇒ ρ needs to increase very fast with E to keep ΔE in reasonable limits (ΔE needs to be compensated by accelerating RF cavities!)

⇒ since $B_{dipole} \propto \frac{E}{\rho}$ but $\Delta E_s \propto \frac{E^4}{\rho}$

⇒ high E circular accelerators have weaker bending magnets

⇒ energy limit for circular e^- accelerators ~ 100 GeV (see LEP with $\rho = 3$ km!)

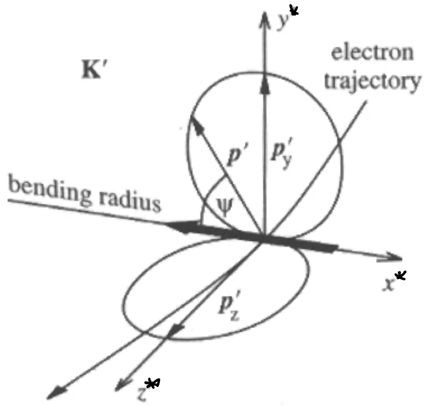
⇒ for $E \gtrsim 100$ GeV: linear accelerators (ILC), or use heavy particles like protons, μ

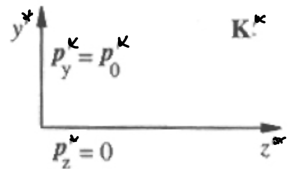
Synchrotron Energy Loss Examples

Table 2.1 A few important circular electron accelerators. L is the circumference of the machine, E the maximum beam energy, R the bending radius, B the field in the bending magnets and ΔE the energy loss per revolution.

accelerator	L [m]	E [GeV]	$\cancel{R}$ ρ [m]	B [T]	ΔE [keV]
BESSY I (Berlin)	62.4	0.80	1.78	1.50	20.3
DELTA (Dortmund)	115	1.50	3.34	1.50	134.1
DORIS II (Hamburg)	288	5.00	12.2	1.37	4.53×10^3
ESRF (Grenoble)	844	6.00	23.4	0.855	4.90×10^3
PETRA (Hamburg)	2304	23.50	195	0.40	1.38×10^5
LEP (Geneva)	27×10^3	70.00	3000	0.078	7.08×10^5

e) Angular distribution for circular accelerators





In co-moving frame:

$$\frac{dP_s^*}{d\Omega^*} \propto \sin^2 \Phi$$

(frame moves along z-direction)

$\rightarrow$ consider photon emitted along y^* axis in co-moving frame:

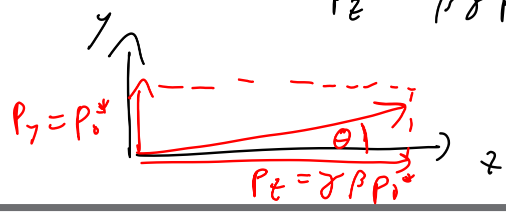
$$\vec{p}^{\,*} = \begin{pmatrix} 0 \\ p_0^* \\ 0 \end{pmatrix} \Rightarrow \text{four momentum vector } p_\mu^* = \begin{pmatrix} p_t^* \\ p_x^* \\ p_y^* \\ p_z^* \end{pmatrix} = \begin{pmatrix} E_\gamma^*/c \\ 0 \\ p_0^* \\ 0 \end{pmatrix}$$

$\Rightarrow$ Lorentz-transformation into lab frame (e^- moves along z)

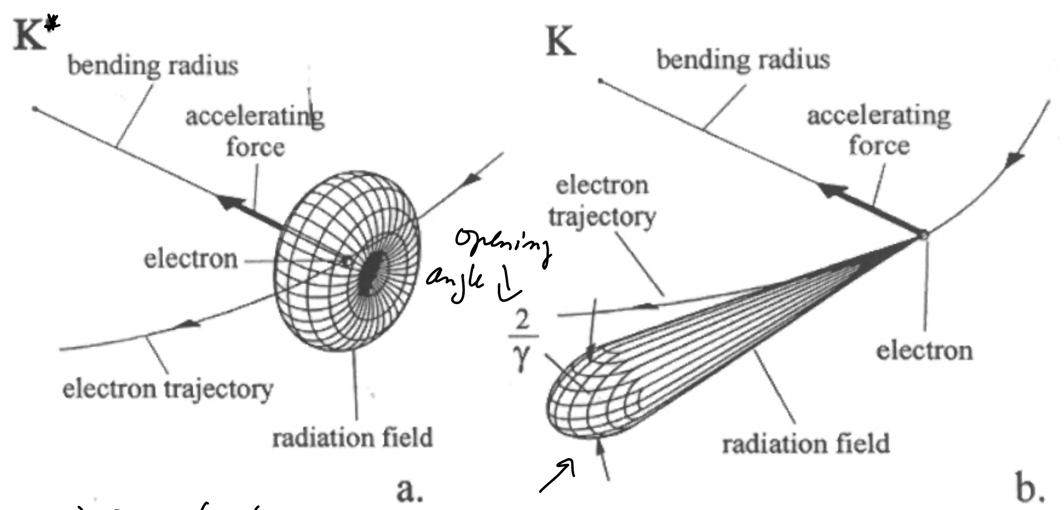
$$p_\mu = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} E_\gamma^*/c \\ 0 \\ p_0^* \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma E_\gamma^*/c \\ 0 \\ p_0^* \\ \gamma\beta \frac{E_\gamma^*}{c} \end{pmatrix}$$

$\Rightarrow \frac{E_\gamma}{E_\gamma^*} = \gamma \Rightarrow$ high energy $\Rightarrow$ high $\omega_{photon} = E_\gamma/\hbar$

$\Rightarrow \tan\theta = \frac{p_y}{p_z} = \frac{p_0^*}{\beta\gamma p_0^*} = \boxed{\frac{1}{\gamma} \simeq \theta}$: angle of photon in lab frame relative to z-axis (forward direction of e^-)



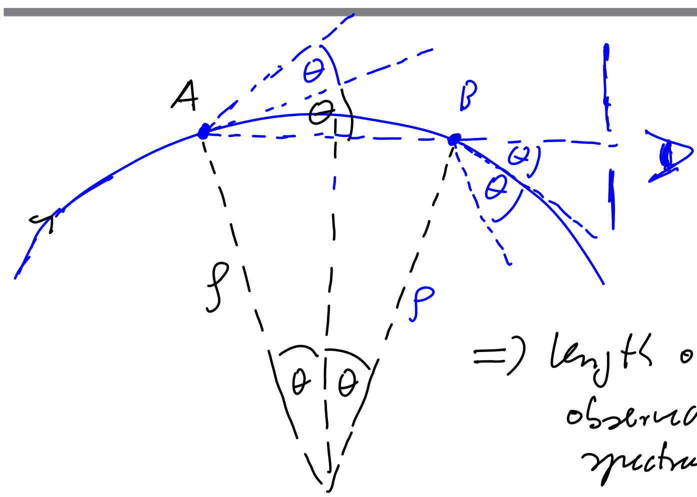
Transformation from COM frame to lab frame

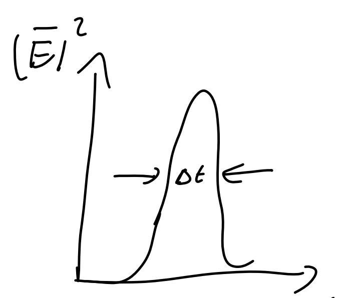


⇒ synchrotron radiation in lab frame has sharply forward peaked distribution!

Example: e^- with $E = 1\,\text{GeV} \Rightarrow \gamma = 1957$
$\Rightarrow \theta \approx 0.03°$

f) Frequency spectrum of synchrotron radiation





⇒ length of radiation pulse Δt observed determines frequency spectrum

⇒ radiation cone angle $\theta = \frac{1}{\gamma}$

⇒ length of pulsed observed:

$$\Delta t = t_{e^-, A \to B} - t_{\gamma, A \to B} = \frac{2\rho\theta}{c\beta} - \frac{2\rho \sin\theta}{c}$$

$$\Rightarrow \Delta t \approx \frac{2\rho}{\gamma c}\Big\{ \underbrace{\frac{1}{\beta} - 1 + \frac{1}{6\gamma^2}}_{\approx \frac{2}{3}\frac{1}{\gamma^2} \text{ for } \beta \approx 1} \Big\} \approx \frac{4\rho}{3\gamma^3 c}$$

$\Rightarrow$ short radiation pulse has broad spectrum with "characteristic frequency"

$$\omega_{char} = \frac{2\pi}{\Delta t} = \frac{3\pi c \gamma^3}{2\rho}$$

$\Rightarrow$ define "critical frequency": ω_c

$$\underline{\omega_c} = \frac{\omega_{char}}{\pi} = \underline{\underline{\frac{3}{2}\frac{c\gamma^3}{\rho}}}$$