Lecture 22

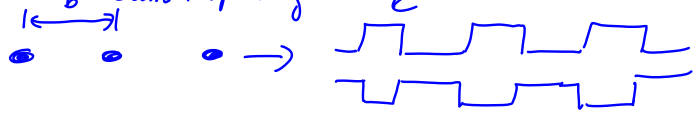
5. RF Systems and Particle Acceleration

5.4 Phase focusing and longitudinal (synchrotron) beam oscillation

(a) Linac

T_b : bunch spacing

accelerating voltage $V(t) = \hat{V} \sin(\omega_{rf} \cdot t)$



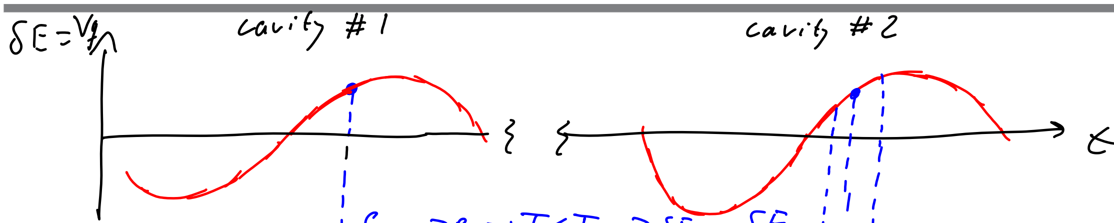
$\Rightarrow$ for synchronization, need:

① $f_{RF} = h \frac{1}{T_b} = h f_b$

integer

② phase focusing (for $v < c$ particle)

for $v \ll c$



for $p > p_0 \Rightarrow T < T_0 \Rightarrow \delta E < \delta E_0$

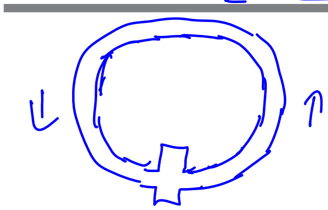
for $p = p_0 \Rightarrow T = T_0 \Rightarrow \delta E = \delta E_0$

for $p < p_0 \Rightarrow T > T_0 \Rightarrow \delta E > \delta E_0$

$\Rightarrow \delta E_0$ = energy gain of reference particle with $\delta = 0$ momentum error, passing cavity at reference phase $\phi = \phi_0$

$\Rightarrow$ for stable operation, need $\boxed{0 < \phi_0 < \frac{\pi}{2}}$

(b) for circular accelerators



$V = \hat{V} \sin(\omega_{RF} t)$

revolution time for ideal, reference particle: $T_0 = \frac{C_0}{v_0}$ ← circumference of ring, ← speed

$\Rightarrow f_0 = \frac{1}{T_0}$

$\Rightarrow$ Energy gain per turn: $\delta E = q \hat{V} \sin \phi$

ϕ: RF accel. phase during particle acceleration : $\phi = \omega t_{pass}$

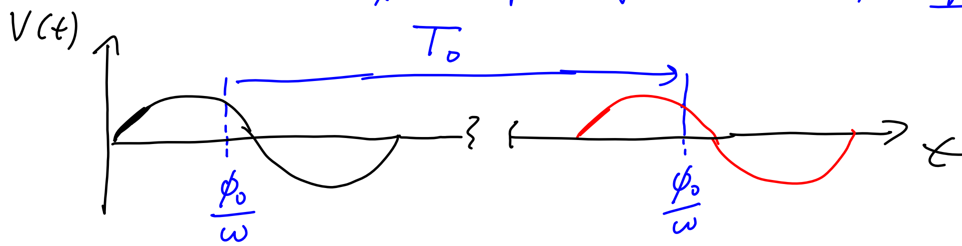
$\Rightarrow$ for synchronization, need:

1) $f_{RF} = h \cdot f_0$

↑ integer, harmonic number

$\Rightarrow$ for protons in synchrotron: $v \ll c \Rightarrow f_0$ increases with energy over time $\Rightarrow f_{RF}$ needs to increase during energy ramp up

2) correct accelerating phase ϕ_0 for reference particle with $\delta = 0$ (same phase for each turn) $\Rightarrow$ phase focusing!



ideal (reference) particle $\Rightarrow$ passes at nominal phase ϕ_0

particle with $\delta \neq 0$ $\Rightarrow$ phase focusing results in longitudinal particle oscillation about nominal phase ϕ_0

$\Rightarrow$ synchrotron oscillations

- revolution time $T = C/v$

  $\Rightarrow$ depends on circumference C and particle speed v

  $\Rightarrow$ both depend on particle momentum!

  $\Rightarrow$ for small changes ΔC, Δv

$$\frac{\Delta T}{T} = \frac{\Delta C}{C} - \frac{\Delta v}{v}$$

$$\frac{\Delta v}{v} = \frac{1}{\gamma^2} \frac{\Delta p}{p} \Leftarrow p = \gamma m_0 v$$

recall momentum compaction factor

$$\alpha \equiv \frac{\Delta C / C}{\Delta p / p} = \frac{1}{C} \oint \frac{D(s)}{\rho(s)} \, ds > 0$$

$$\Rightarrow \quad \frac{\Delta T}{T} = \left(\alpha - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}$$

$\Rightarrow$ define "transition energy"

$$E_{tr} = \gamma_{tr}\, m_0 c^2 \quad \text{with} \quad \frac{1}{\gamma_{tr}^2} = \alpha$$

$$\Rightarrow \quad \frac{\Delta T}{T} = 0 \quad \text{at} \quad E = E_{tr}$$

γ_{tr} is defined by the accelerator optics

$$\boxed{\frac{\Delta T}{T} = \left(\frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}\right)\frac{\Delta p}{p}}$$

$\Rightarrow$ two distinct regions:

- $E < E_{tr} \Rightarrow \gamma < \gamma_{tr} \Rightarrow \Delta T < 0$ for $\Delta p > 0$

  $\Rightarrow$ increase in particle energy reduces revolution time, since increase in particle speed dominates over increase in circumference/path length

- $E > E_{tr} \Rightarrow \gamma > \gamma_{tr} \Rightarrow \Delta T > 0$ for $\Delta p > 0$

  $\Rightarrow$ increase in particle energy increases revolution time since increase in path length dominates over increase in particle speed

Note: – for electrons: $v \approx c \Rightarrow E > E_{tr}$ always

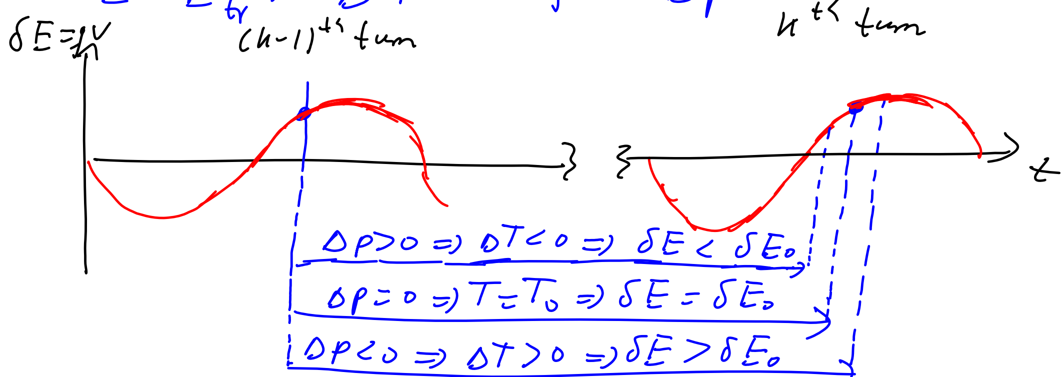
– for protons/ions in synchrotron:

often: $E_{min/injection} < E_{tr} < E_{final}$

$\Rightarrow$ energy passes through E_{tr} during ramp up of energy!

$\Rightarrow$ depending on region, different nominal accelerating phase ϕ_0 of ideal particle is required for stable operation

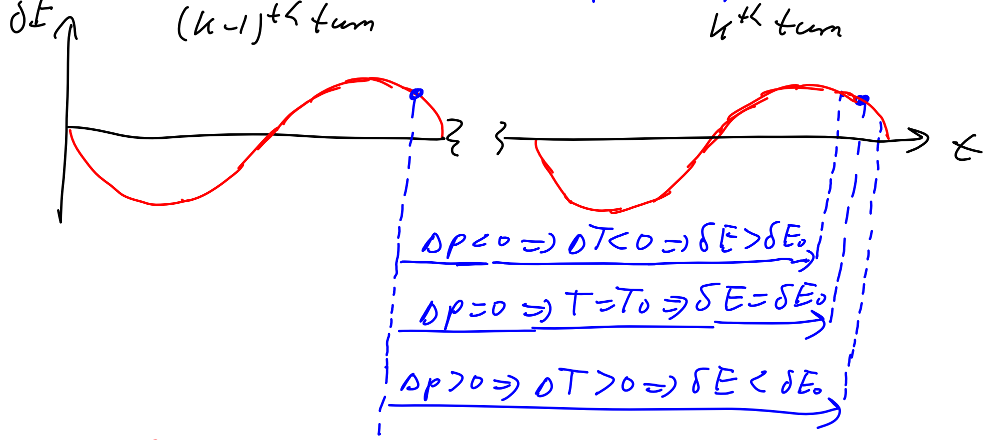
• $E < E_{tr}$: $\Delta T < 0$ for $\Delta p > 0$



$\Rightarrow$ "restoring force" for $\boxed{0 < \phi_0 < \frac{\pi}{2}}$

$\Rightarrow$ synchrotron oscillations about $\phi = \phi_0$ for particle with $\delta \neq 0$

- $E > E_{tr}$: $\Delta T > 0$ for $\Delta p > 0$



$\Rightarrow$ for stable operation during acceleration, need

$$\boxed{\frac{\pi}{2} < \phi_0 < \pi}$$

- differential equation for longitudinal synchrotron oscill.:
  - Energy gain in RF cavity from turn $k-1$ to turn k: δE

$$\delta E = (E_k - E_{k-1}) = q \hat{V} \sin\phi$$

for ideal particle:

$$\delta E_0 = (E_{k,0} - E_{k-1,0}) = q \hat{V} \sin\phi_0$$

  - Energy difference between arbitrary particle and ideal particle: ΔE

$$\Rightarrow \delta E - \delta E_0 = q \hat{V} (\sin\phi - \sin\phi_0) = \Delta E_k - \Delta E_{k-1}$$

- assume that change of ΔE per turn is small
  $\Rightarrow$ rate of change of ΔE

$$\frac{d}{dt}(\Delta E) \approx \frac{\Delta E_k - \Delta E_{k-1}}{T_0}$$

$T_0 \leftarrow$ revolution time $T_0 = 2\pi/\omega_0$

$$\Rightarrow \frac{1}{\omega_0}\frac{d(\Delta E)}{dt} = \frac{q\hat{V}}{2\pi}(\sin\phi - \sin\phi_0) \qquad (1)$$

$\phi_{k,0} = \phi_{k-1,0} = \phi_0 = \text{const}$

$\phi_k - \phi_{k-1} = \omega_{RF}\cdot\Delta T = h\,\omega_0\,\Delta T$

$\Delta T = T - T_0$ = revolution time difference compared to T_0 of ideal particle

- rate of change of accel. phase:

$$\frac{\Delta\phi_{\text{per turn}}}{T_0} = \frac{\phi_k - \phi_{k-1}}{T_0} \approx \frac{d\phi}{dt}$$

$\phi_k - \phi_{k-1} = h\,\omega_0\,\Delta T$

$$\Rightarrow \frac{d\phi}{dt} \approx h\,\omega_0\frac{\Delta T}{T_0} = h\,\omega_0\,\eta\frac{\Delta p}{p}$$

with $\eta = \left(\frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2}\right)$

$\Rightarrow$ since $\frac{dE}{dp} = \beta^2\frac{E}{p}$

$$\frac{d\phi}{dt} \approx h\,\omega_0\,\eta\frac{1}{\beta_0^2}\frac{\Delta E}{E_0} \qquad (2) \quad \Big|\ \frac{d}{dt}$$

$\Rightarrow$ with (1) for $\frac{d(\Delta E)}{dt}$

$$\boxed{\frac{E_0 \beta_0^2}{\omega_0^2 h \eta} \frac{d^2\phi}{dt^2} = \frac{q\hat{V}}{2\pi}(\sin\phi - \sin\phi_0)} \quad (3)$$

diff.-eqn. for synchrotron oscillations $\phi(t)$ in adiabatic approximation: assume that $E_0, \omega_0, \beta_0, \eta$ are changing only slowly, i.e. are $\approx$ const within time scale T_0

$\Rightarrow$ for small deviations $\Delta\phi = \phi - \phi_0$ from ideal phase ϕ_0: $\Rightarrow$ simple harmonic oscillations

$$\sin\phi = \sin(\phi_0 + \Delta\phi) = \sin\phi_0 \underbrace{\cos\Delta\phi}_{\approx 1} + \cos\phi_0 \underbrace{\sin\Delta\phi}_{\Delta\phi}$$

$$\approx \sin\phi_0 + \cos\phi_0 \Delta\phi$$

also: $\frac{d^2(\Delta\phi)}{dt^2} = \frac{d^2\phi}{dt}$, since $\phi_0 = \text{const}$

$\Rightarrow$ this gives for small $\Delta\phi$:

$$\frac{d^2(\Delta\phi)}{dt} + \Omega^2 \Delta\phi = 0 \quad \text{with } \Omega^2 = -\eta\cos\phi_0 \frac{q\hat{V}}{2\pi}\frac{\omega_0^2 h}{E_0\beta_0^2}$$

$\Rightarrow$ for $\Omega^2 > 0$: simple harmonic oscillations with frequency Ω : <u>synchrotron frequency</u>

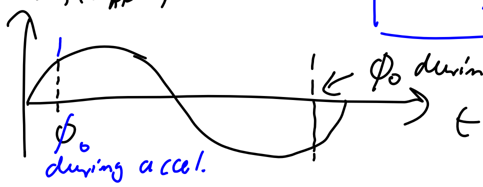
$\Rightarrow$ if $\Omega^2 < 0$: exponential solutions $\Rightarrow$ instability

$\Rightarrow$ need $\Omega^2 > 0$ for stable operation

- $E < E_{tr}$ case: $\eta = \left(\frac{1}{\gamma_{tr}^2} - \frac{1}{\gamma^2} \right) < 0$

$\Rightarrow$ need $\cos \phi_0 > 0$

$V(t) = \hat{V} \sin(\omega_{RF} t)$ $\Rightarrow$ need $\boxed{0 < \phi_0 < \frac{\pi}{2}}$ for stability during acceleration



$\leftarrow$ ϕ_0 during decelleration $\left(\frac{3}{2}\pi < \phi_0 < 2\pi \right)$