Lecture 13

4. Beam optics in circular accelerators

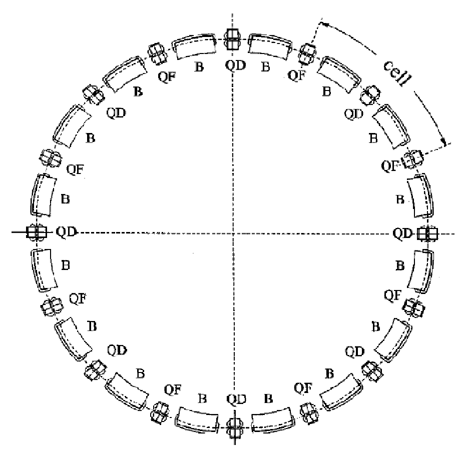
4.4 FODO cell

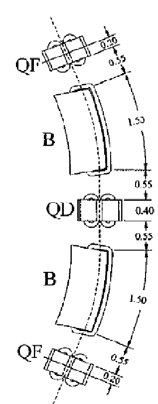
4.5 The periodic (closed) orbit

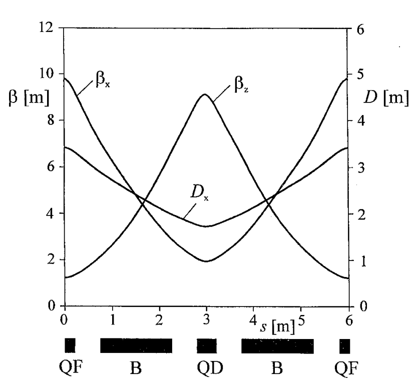
4.6 Closed orbit for $\Delta p \neq 0$

4.4 The FODO cell

Alternating gradients allow focusing in both transverse plains. Therefore focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.

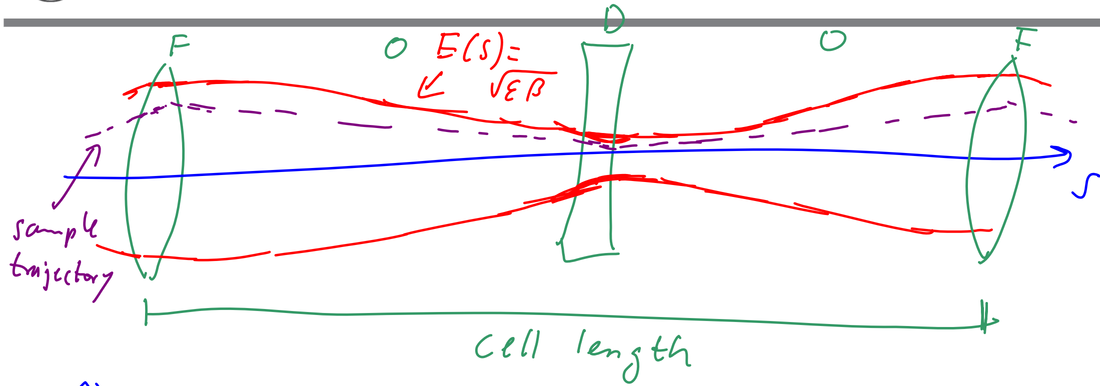






The periodic beta function and dispersion for each FODO is also periodic for an accelerator section that consists of many FODO cells. Often large sections of an accelerator consist of FODOs.

Example: Thin lens FODO cell



1) in "thin lens" approximation with drift between quadrupoles: thin lens: $\left(\frac{1}{f}\right) = kl$

$\Rightarrow$ for horizontal plane:

$$\underline{M}_{FODO} = Q\left(-\frac{kl}{2}\right) D\left(\frac{L}{2}\right) Q\left(\frac{kl}{2}\right) Q\left(\frac{kl}{2}\right) D\left(\frac{L}{2}\right) Q\left(-\frac{kl}{2}\right)$$

$$\Rightarrow \underline{M}_{FODO} = \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & L/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{kl}{2}\frac{L}{2} & \frac{L}{2} \\ -\left(\frac{kl}{2}\right)^2 \frac{L}{2} & 1 - \frac{kl}{2}\frac{L}{2} \end{pmatrix} \begin{pmatrix} 1 - \frac{kl}{2}\frac{L}{2} & \frac{L}{2} \\ -\left(\frac{kl}{2}\right)^2 \frac{L}{2} & 1 + \frac{kl}{2}\frac{L}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2\xi^2 & L(1+\xi) \\ -\left(\frac{kl}{2}\right)^2 L(1-\xi) & 1 - 2\xi^2 \end{pmatrix} \quad \text{with } \xi \equiv \frac{kl}{2}\frac{L}{2}$$

$\Rightarrow$ ① $\cos\mu = \frac{1}{2}\,\mathrm{trace}\,\underline{M}_{FODO} = 1 - 2\xi^2$

$\Rightarrow$ with $\cos\mu = 1 - 2\sin^2\mu/2$

$\Rightarrow$ $\boxed{\left|\sin\left(\frac{\mu}{2}\right)\right| = |\xi|}$ for FODO cell

$\Rightarrow$ stability requires: $|\mathrm{trace}\,\underline{M}_{FODO}| < 2$

$\Rightarrow$ need $\xi < 1$ $\Rightarrow$ $\frac{kl}{2}\frac{L}{2} < 1$

$\Rightarrow$ need focal length $\boxed{f = \frac{1}{kl} > \frac{L}{4}}$

$\Rightarrow$ gives upper limit for focal strength
with betatron phase advance per FODO cell: $0° < \mu < 180°$

② β-function at FODO cell entrance/exit:
(at center of focusing quad)

$$\hat{\beta}_x = M_{FODO,12}\,\frac{1}{\sin\mu} = \frac{L(1+\xi)}{\sin\mu}$$

$$\Rightarrow \boxed{\hat{\beta}_x = L\,\frac{1+\sin(\mu/2)}{\sin\mu}}$$

③ α-function at FODO cell entrance/exit:
(at center of focusing quad)

$$\alpha = \left(M_{FODO,11} - M_{FODO,22}\right)\frac{1}{2\sin\mu} = 0$$

as expected from symmetry

④ β- function at FODO cell center (center of defocusing quadrupole):

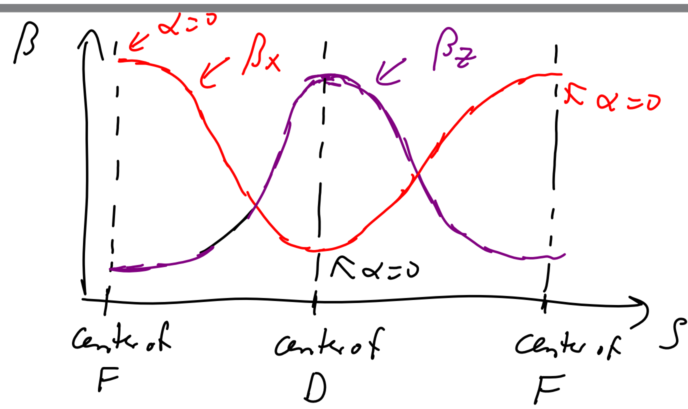
use $\check{\beta} = C^2 \hat{\beta} + S^2 \frac{1}{\hat{\beta}}$ since $\alpha_0 = 0$

from transport matrix of first half of FODO cell: $Q\left(\frac{k\ell}{2}\right) D\left(\frac{L}{2}\right) Q\left(-\frac{k\ell}{2}\right)$

have $C_{1/2\,FODO} = 1 - \xi$ $\qquad S_{1/2\,FODO} = L/2$

$$\Rightarrow \check{\beta} = (1-\xi)\hat{\beta} + \frac{L^2}{4}\frac{1}{\hat{\beta}}$$

$$\Rightarrow \boxed{\check{\beta}_x = L\,\frac{1-\sin \mu/2}{\sin \mu}}$$



4.5 The periodic (closed) orbit

so far: design orbit of circular machine

$\rightsquigarrow$ closed curve $\Rightarrow$ periodic

$\vec{u}(s) = 0$ $\Big\{$ $\rightsquigarrow$ particle with $u_0 = 0$, $u_0' = 0$, $\delta = 0$ travels on design orbit

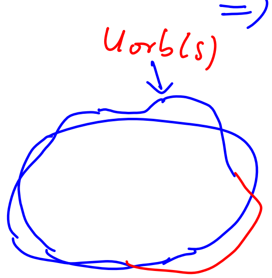
$\vec{u}_\beta(s) = \underline{M}\,\vec{u}_\beta(0)$ $\Big\{$ $\rightsquigarrow$ particles with $u_0 \neq 0$ and/or $u_0' = 0$, but $\delta = 0$ will conduct betatron oscillations about design orbit

Note: path of particles around ring does not close onto itself since the ν-value is non-integer!

now: consider distorted orbit:

$\rightarrow$ e.g. from extra kick from dipole field error, particle with momentum error (i.e. $\delta \neq 0$) ...

$\Rightarrow$ The design orbit is no longer a possible trajectory for a particle!



$\Rightarrow$ new, distorted closed orbit

note: for stable beam operation, this must be still a periodic, closed orbit!

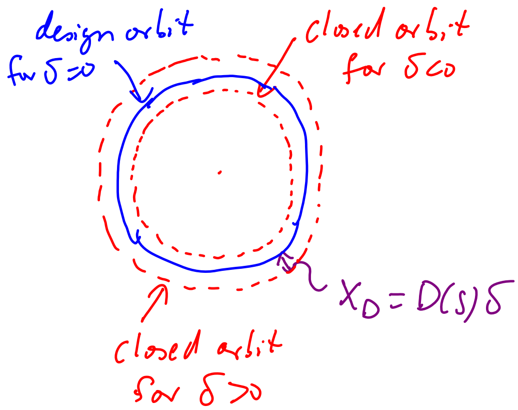
$\vec{u} = \vec{u}_{orb}(s)$

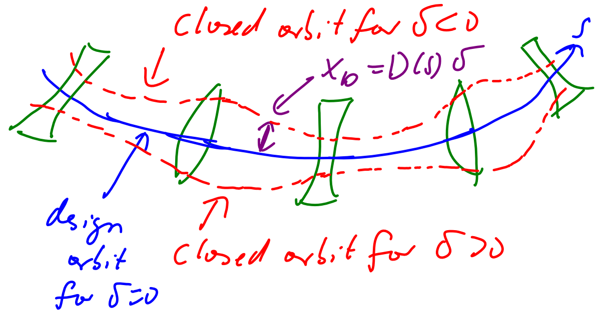
with $\vec{u}_{orb}(s+L) = \vec{u}_{orb}(s)$

⇒ Example: Particle with momentum error.

weakly focusing circular accelerator

strongly focusing circular accelerator





⇒ closed dispersion orbit defined by

$$x_D(s) = \underbrace{D(s)}_{\text{closed, periodic dispersion function}}\,\delta$$

$$D(s+L) = D(s)$$

$$D'(s+L) = D'(s)$$

⇒ particles with $u_\beta(0) = 0$ and $u_\beta'(0) = 0$ move on closed distorted orbit

i.e. $\vec{u}(s) = \vec{u}_{orbit}(s)$

$\Rightarrow$ particle with $u_\beta(0) \neq 0$ and/or $u_\beta'(0) \neq 0$ perform betatron oscillations about the distorted closed orbit, and not around the original design orbit!

$$\Rightarrow \quad \vec{u}(s) = \underbrace{\vec{u}_{orbit}(s)}_{\substack{\text{particular,}\\ \text{periodic solution}\\ \text{of the inhomog.}\\ \text{equ. of motion}}} + \underbrace{\vec{u}_\beta(s)}_{\substack{\text{solution of}\\ \text{the homogeneous}\\ \text{equ. of motion}}}$$

$\vec{u}_\beta(s) = \underline{M}\, \vec{u}_\beta(0)$

as before!

equ. of motion: $\quad u'' + K(s)\, u = \underset{\substack{\uparrow\\ \text{perturbation term}\\ \text{from magnetic field}\\ \text{error, particle error,...}}}{P(s)}$

$\rightsquigarrow$ $u_{orb}(s)$ is a periodic solution of this inhomogeneous equation of motion, i.e. need to find solution with $\vec{u}_{orb}(s+L) = \vec{u}_{orb}(s)$

$\rightsquigarrow$ start with general solution (from before)

$u(s) = a\, C(s) + b\, S(s) + P(s)$

with special (non-periodic!) solution of the inhomog. equation:

$$P(s) = S(s)\int_0^s p(\tilde{s})C(\tilde{s})d\tilde{s} - C(s)\int_0^s p(\tilde{s})S(\tilde{s})d\tilde{s}$$

$\rightsquigarrow$ need to find constants a and b to find <u>periodic</u> solution u_{orb}:

at $s_0 = 0$: $\quad C(s_0) = 1 \qquad C'(s_0) = 0$

$\qquad S(s_0) = 0 \qquad S'(s_0) = 1$

$\qquad P(s_0) = 0 \qquad P'(s_0) = 0$

need $u_{orb}(s_0) = u_{orb}(s_0 + L)$

gives: $a + 0 + 0 = a\,C(L) + b\,S(L) + P(L)$ ①

also need: $u'_{orb}(0) = u'_{orb}(L)$

gives: $0 + b + 0 = a\,C'(L) + b\,S'(L) + P'(L)$ ②

$\Rightarrow$ solve ② for b, insert into ①

$$\Rightarrow a\left(1 - C(L) - \frac{S(L)\,C'(L)}{1 - S'(L)}\right) = \frac{S(L)\,P'(L)}{1 - S'(L)} + P(L)$$

$$\Rightarrow a = \frac{S(L)\,P'(L) - \{S'(L) - 1\}\,P(L)}{\{C(L) - 1\}\{S'(L) - 1\} - S(L)\,C'(L)}$$

~) The denominator is:

$$\underline{\text{denom}} = 1 + \{ \underbrace{C(L)\,S'(L) - S(L)\,C'(L)}_{\det \underline{M} = 1} \} - \{ \underbrace{C(L) + S'(L)}_{\substack{\text{trace } \underline{M} \\ = 2\cos\mu}} \}$$

$$= 2 - 2\cos\mu = \underline{4\sin^2(\mu/2)}$$

$\uparrow$ betatron phase advance per full turn $= 2\pi \cdot$ tune

~) The numerator is:

$$\text{num.} = S(L)\left\{ S'(L)\int_0^L p(\tilde{s})\,C'(\tilde{s})\,d\tilde{s} - C'(L)\int_0^L p(\tilde{s})\,S(\tilde{s})\,d\tilde{s} \right\}$$

$$- (S'(L) - 1)\left\{ S(L)\int_0^L p(\tilde{s})\,C'(\tilde{s})\,d\tilde{s} - C(L)\int_0^L p(\tilde{s})\,S'(\tilde{s})\,d\tilde{s} \right\}$$

$$= \{ \underbrace{C(L)\,S'(L) - S(L)\,C'(L)}_{\det M = 1} \} \int_0^L p\,S\,d\tilde{s} - C(L)\int_0^L p\,S\,d\tilde{s}$$

$$+ S(L)\int_0^L p\,C'\,d\tilde{s}$$

for periodic structure:

$$\underline{M}(0 \to L) = \begin{pmatrix} \cos\mu + \alpha_0 \sin\mu & \beta_0 \sin\mu \\ -\gamma_0 \sin\mu_0 & \cos\mu - \alpha_0 \sin\mu \end{pmatrix}$$

$$\Rightarrow \quad C'(L) = \cos\mu + \alpha_0 \sin\mu$$

$$S'(L) = \beta_0 \sin\mu$$

$$\Rightarrow \text{num.} = (1 - \cos\mu - \alpha_0 \sin\mu) \int_0^L p(\tilde{s})\, S(\tilde{s})\, d\tilde{s}$$

$$+ \beta_0 \sin\mu \int_0^L p(\tilde{s})\, C'(\tilde{s})\, d\tilde{s}$$

$\Rightarrow$ from general transport matrix in terms of Twiss parameters:

$$C'(s) = \sqrt{\beta/\beta_0}\, [\cos(\psi - \psi_0) + \alpha_0 \sin(\psi - \psi_0)]$$

$$S' = \sqrt{\beta\beta_0}\, \sin(\psi(s) - \psi_0)$$

$$\Rightarrow \text{num.} = (1 - \cos\mu - \alpha_0 \sin\mu) \sqrt{\beta_0} \int_0^L p(\tilde{s}) \sqrt{\beta(\tilde{s})} \sin(\psi(\tilde{s}) - \psi_0)\, d\tilde{s}$$

$$+ \sqrt{\beta_0} \sin\mu \int_0^L p(\tilde{s}) \sqrt{\beta(\tilde{s})} [\cos(\psi(\tilde{s}) - \psi_0) + \alpha_0 \sin(\psi(\tilde{s}) - \psi_0)]\, d\tilde{s}$$

$$= \sqrt{\beta_0}\, 2 \sin\left(\frac{\mu}{2}\right) \int_0^L \sqrt{\beta(\tilde{s})}\, p(\tilde{s}) \cos\left\{\psi(\tilde{s}) - \psi_0 - \frac{\mu}{2}\right\} d\tilde{s}$$

now: $u_{orb}(0) = a$, point $s_0 = 0$ was chosen arbitrarily!

$$\Rightarrow u_{orb}(s) = \frac{\sqrt{\beta(s)}}{2\sin\left(\frac{\mu}{2}\right)} \oint_{\text{full turn}} p(\tilde{s})\sqrt{\beta(\tilde{s})}\cos\left\{|\psi(\tilde{s}) - \psi(s)| - \frac{\mu}{2}\right\} d\tilde{s}$$

note: for $\mu = N 2\pi \Rightarrow$ instability!

(i.e. for tune ν = integer)