Lecture 12

3. Linear transverse beam optics

- **3.10 Transport matrix from the Twiss parameters**
- **3.11 Matching of beam optics**

4. Beam optics in circular accelerators

- **4.1 Hill's equation and periodic sections**
- **4.2 Stability criterion**
- **4.3 Tune**

3.10 Transport matrix from the Twiss parameters

have $\alpha, \beta, \gamma, \psi$ at the beginning and end of magnet structure ⇒ get $\underline{M}$ from this

→ initial conditions at s_0

$$u(0) = u_0 \qquad u(0)' = u_0' $$
$$\beta(0) = \beta_0 \qquad \alpha(0) = \alpha_0 \qquad \psi(0) = 0$$

at s_0:
⇒
$$\begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin \phi_0 \\ \cos \phi_0 \end{pmatrix}$$

$\underline{\text{at } s:}$
$$\begin{pmatrix} u \\ u' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s) + \phi_0) \\ \cos(\psi(s) + \phi_0) \end{pmatrix}$$
$$= \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta} \end{pmatrix} \begin{pmatrix} \cos \psi(s) & \sin \psi(s) \\ -\sin \psi(s) & \cos \psi(s) \end{pmatrix} \begin{pmatrix} \sin \phi_0 \\ \cos \phi_0 \end{pmatrix} = M \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}$$

$$\Rightarrow \sqrt{2J}\begin{pmatrix}\sqrt{\beta} & 0\\ -\alpha/\sqrt{\beta} & \frac{1}{\sqrt{\beta}}\end{pmatrix}\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}\sin\phi_0\\ \cos\phi_0\end{pmatrix}$$

$$= \underline{M}\,\sqrt{2J}\begin{pmatrix}\sqrt{\beta_0} & 0\\ -\alpha_0/\sqrt{\beta_0} & 1/\sqrt{\beta_0}\end{pmatrix}\begin{pmatrix}\sin\phi\\ \cos\phi_0\end{pmatrix}$$

$$\Rightarrow M\begin{pmatrix}\sqrt{\beta_0} & 0\\ -\alpha_0/\sqrt{\beta_0} & 1/\sqrt{\beta_0}\end{pmatrix} = \begin{pmatrix}\sqrt{\beta} & 0\\ -\alpha/\sqrt{\beta} & 1/\sqrt{\beta}\end{pmatrix}\begin{pmatrix}\cos\psi(s) & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}$$

$$\Rightarrow M = \begin{pmatrix}\sqrt{\beta} & 0\\ -\alpha/\sqrt{\beta} & \frac{1}{\sqrt{\beta}}\end{pmatrix}\begin{pmatrix}\cos\psi & \sin\psi\\ -\sin\psi & \cos\psi\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{\beta_0}} & 0\\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0}\end{pmatrix}$$

$$\Rightarrow$$

$$\underline{M} = \begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}\left[\cos\psi + \alpha_0\sin\psi\right] & \sqrt{\beta_0\beta}\,\sin\psi\\ \frac{(\alpha_0-\alpha)\cos\psi - (1+\alpha\alpha_0)\sin\psi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}\left[\cos\psi - \alpha\sin\psi\right]\end{pmatrix}$$

3.11 Matching of beam optics

magnet structure

$$\begin{pmatrix}\alpha_0\\ \beta_0\\ \gamma_0\end{pmatrix} \qquad k_1 \quad k_2 \quad k_3 \quad \cdots \quad k_n \qquad \begin{pmatrix}\alpha_E\\ \beta_E\\ \gamma_E\end{pmatrix}$$

initial values of optical functions fixed by pre-accelerator, or initial beam distribution

desired values at end of magnet structure

$$\Rightarrow \quad \left.\begin{aligned} \alpha(s) &= f_1(k_1, \ldots k_n) \\ \beta(s) &= f_2(k_1, \ldots k_n) \\ \gamma(s) &= f_3(k_1, \ldots k_n) \\ \psi(s) &= f_4(k_1, \ldots k_n) \\ D(s) &= f_5(k_1, \ldots k_n) \end{aligned}\right\}$$

use numerical methods to find solution $k_1, \ldots k_n$ that gives desired final values at end of magnet structure

4. Beam optics in circular accelerators

4.1 Hill's equation and periodic sections

4.2 Stability criterion

4.3 Tune

4.1 Hill's equation and periodic sections

- particle of design momentum $p = p_0$ (i.e. $\delta = 0$) in periodic magnet structure

→ Hill's equation:

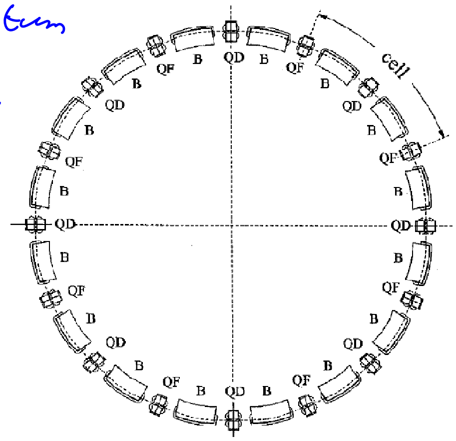
$$u''(s) + \mathcal{K}(s)\,u(s) = 0$$

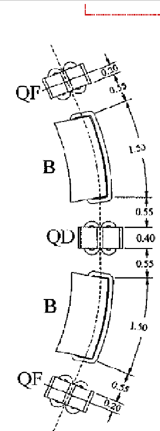
where $\mathcal{K}(s)$ is a periodic function:

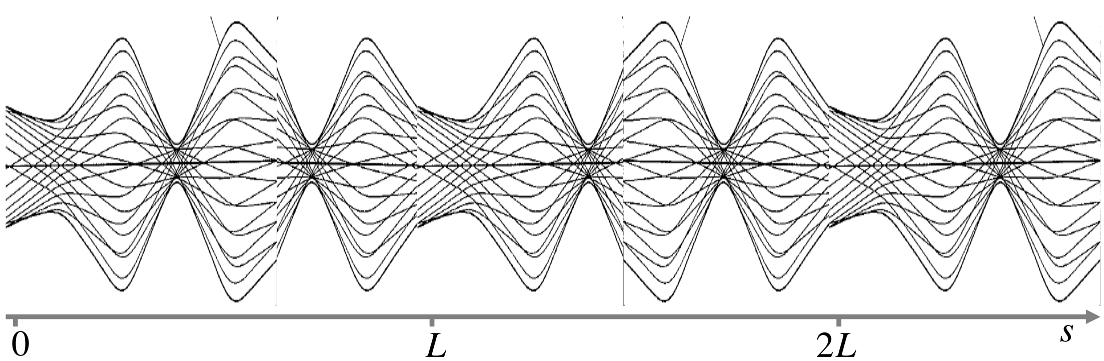
$$\mathcal{K}(s+L) = \mathcal{K}(s)$$

with period L:

$$L = \begin{cases} \text{– circumference of a circular accelerator} \\ \text{– identical "cells" in a periodic section (e.g. FODO cells)} \end{cases}$$







→ solution: $u(s) = \sqrt{\varepsilon}\,\sqrt{\beta(s)}\,\sin(\psi(s) + \phi)$

Twiss parameter α, β, γ must be also periodic, with the same period as $\mathcal{K}(s)$

$$\Rightarrow \beta(s+L) = \beta(s), \ldots$$

⇒ beam envelope $E(s) = \sqrt{\varepsilon\beta}$ / particle distribution $\rho(u, u', s) = \rho(u, u', s+L)$ are periodic for stable operation

Note: $\psi(s) = \int_0^s \frac{1}{\beta(\tilde{s})} d\tilde{s}$, $\underline{M}(0 \to s)$

and individual particle trajectories are <u>not</u> periodic!

⇝ for particle trajectories: $\vec{u} = \begin{pmatrix} u \\ u' \end{pmatrix}$

$$\vec{u}(s) = \underline{M}(0 \to s)\, \vec{u}_0$$

$$\vec{u}(s+L) = \underbrace{\underline{M}(s \to s+L)}_{\equiv \underline{M}_p(s)}\, \vec{u}(s)$$

⇒ therefore $\vec{u}(s+nL) = (\underline{M}_p(s))^n\, \vec{u}(s)$

from before:

$$\underline{M}(s_0 \to s) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta\beta_0}\, \sin\psi \\ \frac{(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0\alpha)\sin\psi}{\sqrt{\beta\beta_0}} & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha\sin\psi] \end{pmatrix}$$

⇒ for transfer matrix of a period from <u>s to $s+L$</u> of a periodic beam line: since $\beta(s+L) = \beta(s)$ $\alpha(s+L) = \alpha(s)$

$$\underline{M}_p(s) = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix}$$

$$\Rightarrow M_p(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin\mu$$

with betatron phase advance per periodic cell / turn

$$\mu \equiv \Psi(s+L) - \Psi(s) = \int_s^{s+L} \frac{1}{\beta(\tilde{s})} d\tilde{s}$$

note: ① $\beta(s) > 0$ allways

② $\boxed{\cos\mu = \frac{1}{2} \text{trace } \underline{M}_p(s)}$

③ periodic Twiss parameters are the solution of the non-linear differential equation

$$\alpha' = \gamma(s)\,\beta(s) - \gamma(s)$$

with periodic boundary conditions:
$\beta(0+L) = \beta(0)$, $\alpha(0+L) = \alpha(0)$

$\Rightarrow$ if $s_0 = 0$ symmetry point:
$$\alpha(0) = -\frac{1}{2}\beta'(0) = 0$$

④ $$\underline{M}_p(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin\mu$$

$$\Leftrightarrow \quad \cos\mu = \frac{1}{2} \text{trace } M_p$$

$$\beta(s) = \underline{M}_{p,12} \frac{1}{\sin\mu}$$

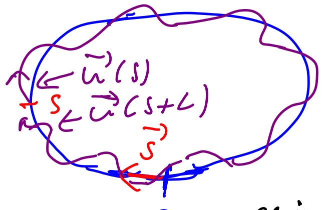
$$\alpha(s) = (M_{p,11} - M_{p,22}) \frac{1}{2\sin\mu}$$

$$\gamma(s) = \frac{1+\alpha^2(s)}{\beta(s)} = -\underline{M}_{p,21} \frac{1}{\sin\mu}$$

4.2 Stability criterion for circular accelerators

$$\vec{u}(s+L) = \underbrace{\underline{M}_p(s)}\,\vec{u}(s)$$

transfer matrix
for full revolution
from s to $s+L$



period L = circumference of ring

↝ for n-turns: $M_{n\,\text{turns}}(s) = \left(\underline{M}_p(s)\right)^n$

↝ condition for stable motion:

elements of matrix $\left(\underline{M}_p(s)\right)^n$ need to remain finite for $n \to \infty$

↝ have: $\underline{M}_p(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin\mu$

$$= \underline{I} \cos\mu + \mathcal{J} \sin\mu$$

with $\mathcal{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$

note: $\det \mathcal{J} = 1$, $\operatorname{trace} \mathcal{J} = 0$, $\mathcal{J}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\underline{I}$

<u>also:</u> $(\underline{I}\cos\mu_1 + \mathcal{J}\sin\mu_1)(\underline{I}\cos\mu_2 + \mathcal{J}\sin\mu_2)$

$$= \underline{I}\cos\mu_1\cos\mu_2 - \underline{I}\sin\mu_1\sin\mu_2 + \underline{I}\mathcal{J}(\cos\mu_1\sin\mu_2 + \sin\mu_1\cos\mu_2)$$

$$= \underline{I}\cos(\mu_1+\mu_2) + \mathcal{J}\sin(\mu_1+\mu_2)$$

therefor: $(M_p(s))^n = I\cos(n\mu) + J\sin(n\mu)$

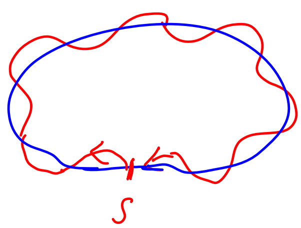
$\Rightarrow$ $(M_p(s))^n$ remains bounded for $n \to \infty$ if and only if μ is real

$$\Rightarrow \boxed{\text{stability} \Leftrightarrow \mu \text{ real} \Leftrightarrow |\text{trace } \underline{M}_p| = |2\cos\mu| \le 2}$$

4.3 Tune ν (also often denoted as Q)

tune $\nu \equiv$ number of betatron oscillations per revolution

$=$ betatron phase advance μ per turn devided by 2π



$$\nu = \frac{\mu}{2\pi} = \frac{\psi(s+L) - \psi(s)}{2\pi}$$

$$= \frac{1}{2\pi} \int_s^{s+L} \frac{1}{\beta(\tilde{s})}\, d\tilde{s}$$

note: – $\beta(s) > 0$ always

– $\nu_{horizontal} \neq \nu_{vertical}$ in general

- The tune of a circular accelerator is a property of the ring and does <u>not</u> depend on the azimuth s!

<u>Proof:</u>

tune does not depend on s!

$$2\cos\mu(s) = \mathrm{trace}\, \underline{M}_p(s \to s+L)$$

trace is not changed by similarity transform.

$$= \mathrm{trace}\left[\underline{M}(0\to s)\,\underline{M}_p(0\to L)\,M^{-1}(0\to s)\right] = \mathrm{trace}\, M_p(0\to L)$$

$$= 2\cos\mu(0)$$

$$\underline{M}(0 \to s+L) = \underline{M}(s \to s+L)\,\underline{M}(0\to s)$$

$$= \underbrace{\underline{M}(L \to s+L)}_{\underline{M}(0\to s)\text{ since periodic}}\,\underline{M}(0\to L)$$

$$\Rightarrow \underline{M}(s\to s+L) = \underline{M}(0\to s)\,\underline{M}(0\to L)\,M^{-1}(0\to s)$$