Lecture 34: 04/17/09

- Particle scattering III
  - Probability current
  - Lot's of scattering
- Barrier penetration - tunneling

Recap

Probability of reflection and transmission:

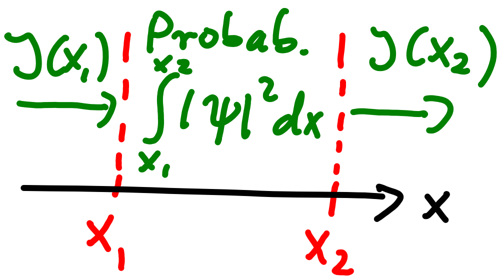
$$R = \frac{|A|^2}{|A_0|^2} = \boxed{\frac{J_{reflected}}{J_{incident}}} \qquad T = 1 - R = \frac{k_2}{k_1}\frac{|B|^2}{|A_0|^2} = \boxed{\frac{J_{transm.}}{J_{incident}}}$$

V_3 Probability Current / Flow

– plane wave: $J = |\psi|^2 \, v_{group} = |\psi|^2 \frac{\hbar k}{m}$

in E & M:

current → Q → current



$$\underbrace{\frac{d}{dt}\int_{x_1}^{x_2} |\Psi(x,t)|^2 dx}_{\text{change in probability between } x_1 \text{ and } x_2} = \underbrace{J_p(x_1,t) - J_p(x_2,t)}_{\text{probability current}}$$

change in probability between x_1 and x_2 = "flowing" in – prob. current "flowing" out

differential form: $\frac{\partial}{\partial t}|\Psi(x,t)|^2 = -\frac{\partial}{\partial x} J_p(x,t)$

$\Rightarrow$ for stationary state: $J_p = \text{constant}$

Note: for stationary states / steady state:

$$\frac{\partial}{\partial t}|\Psi|^2 = 0 \text{ for all } x, t \Rightarrow \frac{\partial}{\partial x} J(x,t) = 0$$

$\Rightarrow J(x,t)$ is constant / conserved!

→ find equation for $J(x,t)$:

- $$\frac{\partial}{\partial t}|\Psi(x,t)|^2 = \frac{\partial}{\partial t}(\Psi^* \Psi) = \Psi^* \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi^*}{\partial t}\Psi$$

  - use time-dep. S.E.:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi(x,t)$$

$$\Rightarrow \quad \frac{\partial \Psi}{\partial t} = -\frac{\hbar}{2mi}\frac{\partial^2 \Psi}{\partial x^2} + \frac{V(x)}{i\hbar}\Psi(x,t)$$

$$\Rightarrow \quad \frac{\partial \Psi^*}{\partial t} = +\frac{\hbar}{2mi}\frac{\partial^2 \Psi^*}{\partial x^2} - \frac{V(x)}{i\hbar}\Psi^*(x,t)$$

$$\Rightarrow \underline{\frac{\partial}{\partial t}|\Psi(x,t)|^2} = \Psi^*\left(-\frac{\hbar}{2mi}\frac{\partial^2\Psi}{\partial x^2} + \frac{V(x)}{i\hbar}\Psi\right)$$

$$+\left(\frac{\hbar^2}{2mi}\frac{\partial^2\Psi^*}{\partial x^2} - \frac{V(x)}{i\hbar}\Psi^*\right)\Psi$$

$V(x)$-terms cancel

$$= -\frac{\hbar}{2mi}\left\{\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right\}$$

$$= -\frac{\partial}{\partial x}\left\{\underbrace{\frac{\hbar}{2mi}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right.}_{J_p(x,t)}\right\}$$

$$= -\frac{\partial}{\partial x}J_p(x,t)$$

with: $J_p(x,t) \equiv \frac{\hbar}{2mi}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)$ prob. current

real number!

$$= \frac{\hbar}{2mi}\Psi^*\frac{\partial\Psi}{\partial x} + \text{complex conj. of first term}$$

3) Try this out for step-up potential:

$$\psi(x) = \begin{cases} A_0 e^{ik_1 x} + A e^{-ik_1 x} & \text{for } x \le 0 \\ B e^{ik_2 x} & \text{for } x > 0 \end{cases}$$

$\Rightarrow$ for $x < 0$ (region I):

$$\underbrace{J_p(x,t)}_{J_{I,\,total}} = \frac{\hbar}{2mi} \underbrace{\left(A_0^* e^{-ik_1 x} + A^* e^{ik_1 x}\right)}_{\psi^*} \underbrace{\left(ik_1 A_0 e^{ik_1 x} - ik_1 A e^{-ik_1 x}\right)}_{\partial \psi / \partial x}$$

$$+ \text{ complex conjugate}$$

$$= \frac{\hbar}{2m}\left(k_1 |A_0|^2 - k_1 |A|^2 - k_1 A_0^* A e^{-2ik_1 x} + k_1 A^* A_0 e^{+2ik_1 x}\right)$$

$$+ \text{ complex conjugate}$$

$$= \frac{\hbar}{2m}\left(2k_1 |A_0|^2 - 2k_1 |A|^2\right) = \underbrace{\frac{\hbar k_1}{m}|A_0|^2 - \frac{\hbar k_1}{m}|A|^2}_{J_{inc} - J_{refl}}$$

as before...

$\Rightarrow$ for $x > 0$ (region II)

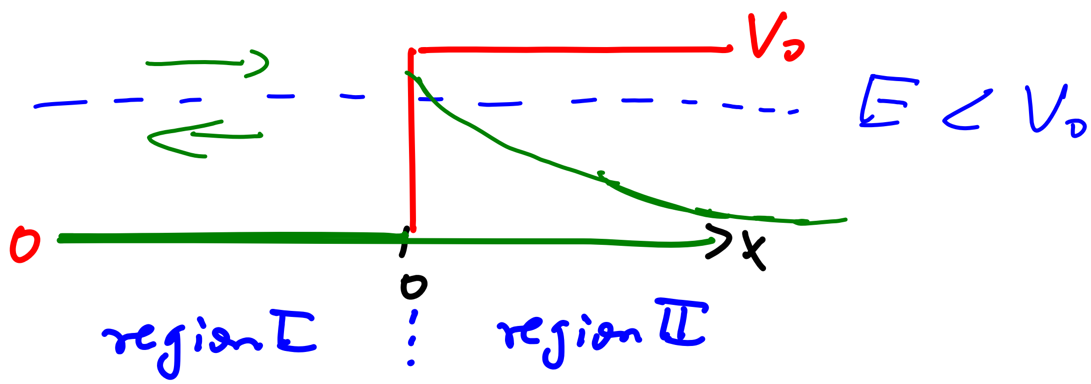
$$J_p(x,t) = \frac{\hbar}{2mi}\Big(\underbrace{B^* e^{-ik_2 x}}_{\psi^*}\Big)\Big(\underbrace{ik_2 B e^{+ik_2 x}}_{\partial\psi/\partial x}\Big)$$

$$+ \text{ complex conjugate}$$

$$= \frac{\hbar}{2m} k_2 |B|^2 \cdot 2 = \underbrace{\underbrace{\frac{\hbar k_2}{m}}_{v_{g,\mathrm{II}}} |B|^2}_{J_{trans.}} \; !$$

as before...

V_4 Step-up Potential (example I) for $0<E<V_0$



→ same math as in example I, but here:

$$k_2 = \frac{\sqrt{2m(E-V_0)}}{\hbar} = i\frac{\sqrt{2m(V_0-E)}}{\hbar} \equiv i\underbrace{\alpha}_{\text{real number}}$$

⇒ stationary state solutions of S.E.:

$$\psi_w(x) = \begin{cases} A_0 e^{ik_1 x} + A e^{-ik_1 x} & \text{for } x \le 0 \\ B e^{ik_2 x} = \underbrace{B e^{-\alpha x}} & \text{for } x > 0 \end{cases}$$

- exponential decay
- no $e^{+\alpha x}$ term!

⇒ A/A_0 and B/A_0 as before (replace k_2 by $i\alpha$)

$\Rightarrow$ reflection coefficient

$$R_W = \left|\frac{A}{A_0}\right|^2 = \frac{J_{refl}}{J_{inc}} = \left|\frac{k_1 - k_2}{k_1 + k_2}\right|^2 = \left|\frac{k_1 - i\alpha}{k_1 + i\alpha}\right|^2$$

$$= \frac{(k_1 - i\alpha)(k_1 + i\alpha)}{(k_1 + i\alpha)(k_1 - i\alpha)} = \underline{\underline{1}} \quad \text{total reflection!}$$

$\Rightarrow$ $T_W = 1 - R_W = 0$ no transmission

Note: - $T = 0$, but wave penetrates slightly into the barrier $\to$ exponential decay to zero for $x \gg 0$

- can not use ~~$J(x>0) = |\Psi|^2 \cdot v_g$~~

(not a plane wave!)

Probability current for $x>0$

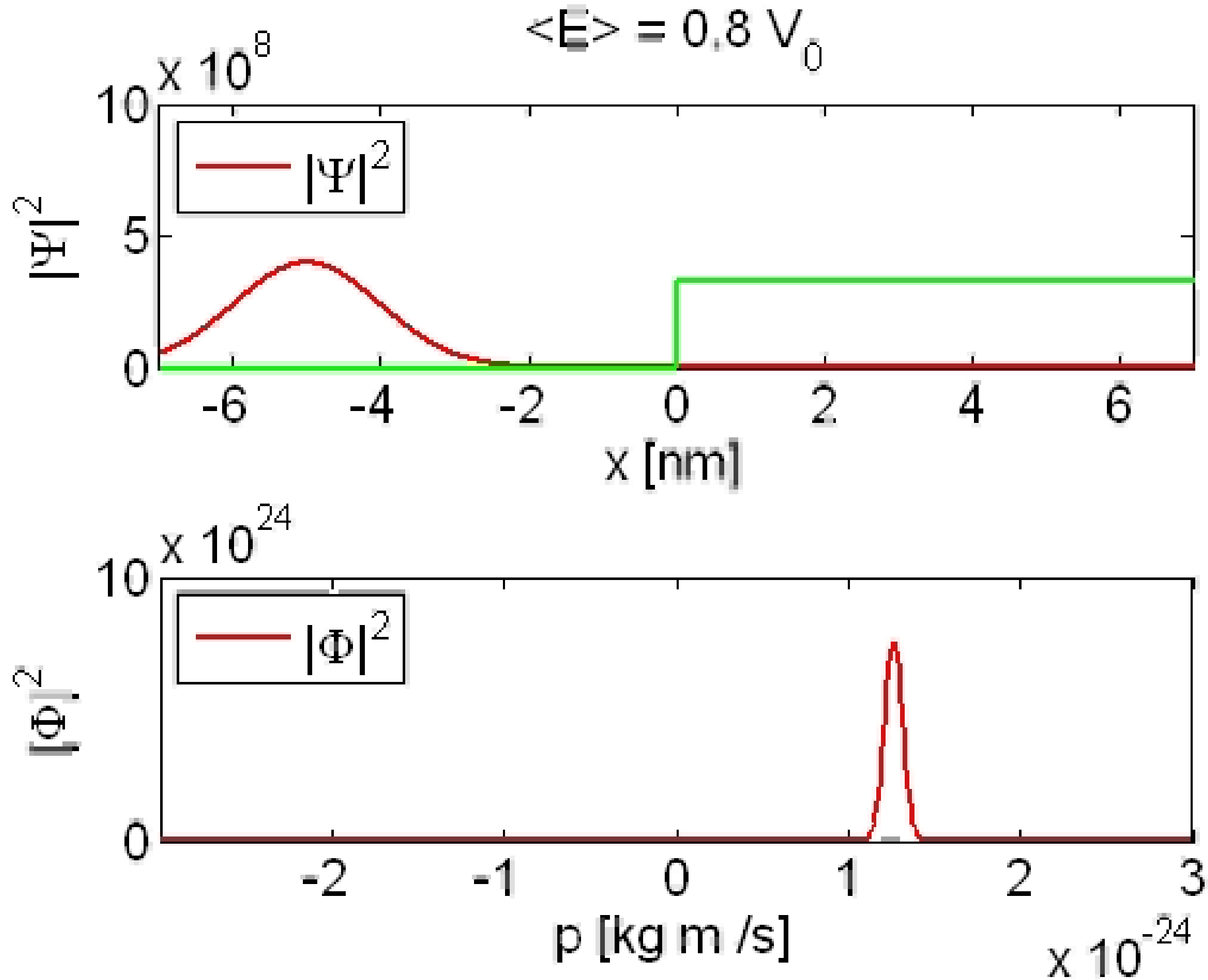
$$\underline{J_p(x>0)} = \frac{\hbar}{2mi}\left(B^* e^{-\alpha x}(-\alpha) B e^{-\alpha x}\right)$$

+ complex conj.

$$= \frac{\hbar}{2m} i \, |B|^2 \alpha \, e^{-2\alpha x} + \text{compl. conj.} = \underline{\underline{0}}$$

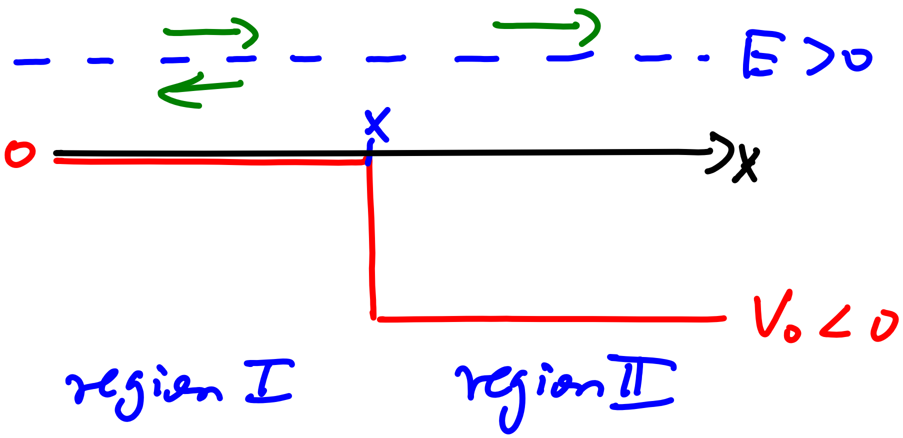
$\Rightarrow$ $T = 0$ $\Rightarrow$ $J_{inc} = -J_{refl.}$

$R = 1 \quad T = 0$



<E> = 0.8 V0
x 10^8
10
5
0
|Ψ|^2
-6
-4
-2
0
2
4
6
x [nm]
x 10^24
10
0
|Φ|^2
-2
-1
0
1
2
3
p [kg m /s]
x 10^-24

V_5 Example II: Step-down Potential



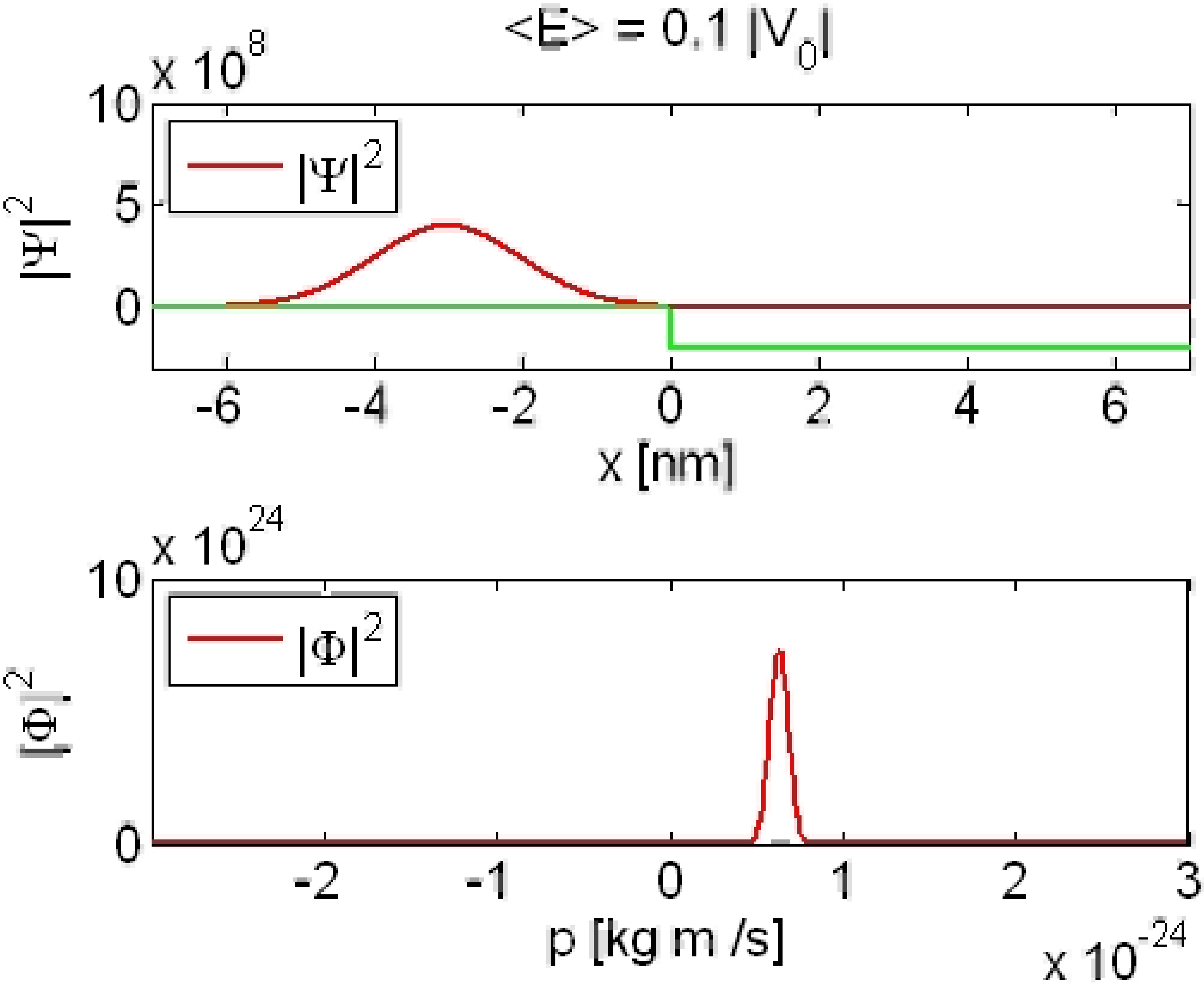
- same math as in example I
- $k_1 < k_2$ here
- see homework

$\Rightarrow$ both, partial reflection and transmission at abrupt change in $V(x)$!

Note: at given particle E

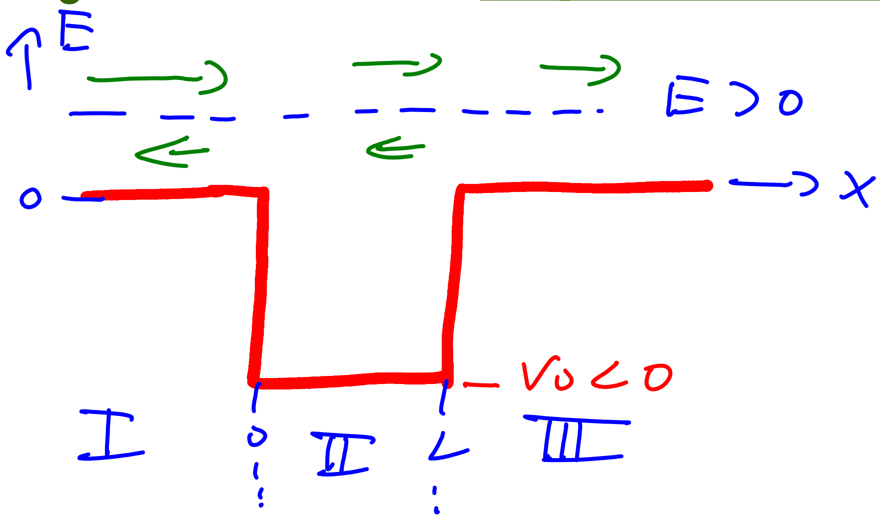
$$R_{\text{from left to right}} = R_{\text{from right to left}}$$

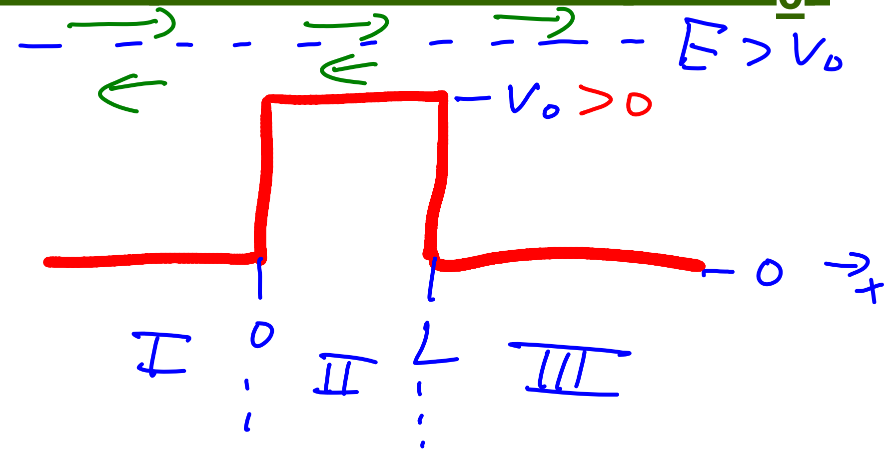
$$\Rightarrow T_{\text{from left to right}} = T_{\text{from right to left}}$$



<E> = 0.1 |V0|
x 10^8
10
5
0
|Ψ|^2
-6
-4
-2
0
2
4
6
x [nm]
x 10^24
10
0
|Φ|^2
-2
-1
0
1
2
3
p [kg m /s]
x 10^-24

V_6 Example III: Square Well and Square Barrier ($E>V_0$)





⇒ 3 regions ⇒ stationary state solutions from time indep. S.E.

$$\Psi_\omega(x) = \begin{cases} A_0 e^{ik_1 x} + A e^{-ik_1 x} & \text{for } x \le 0 \\ B e^{ik_2 x} + C e^{-ik_2 x} & 0 \le x \le L \\ D e^{ik_1 x} & x > L \end{cases}$$

$k_3 = k_1$

apply boundary conditions at $x=0$ <u>and</u> $x=L$ (see Lab)

<u>find:</u>

$$\frac{D}{A_0} = \frac{4k_1k_2}{\left[(k_2+k_1)^2 e^{-ik_2L} - (k_2-k_1)^2 e^{ik_2L}\right] e^{ik_1L}}$$

→ for $E \gg |V_0| \Rightarrow k_1 \approx k_2 \Rightarrow \underline{\underline{T \approx 1}}$

→ if $k_2 L = \overset{\text{integer}}{n}\,\pi \Rightarrow \frac{2\pi}{\lambda_2} L = n\pi \Rightarrow L = n\,\frac{\lambda_2}{2}$

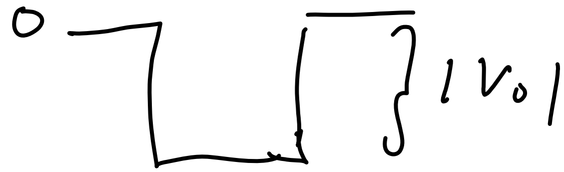
$$\Rightarrow e^{\pm i k_2 L} = e^{\pm i n \pi} = \pm 1$$

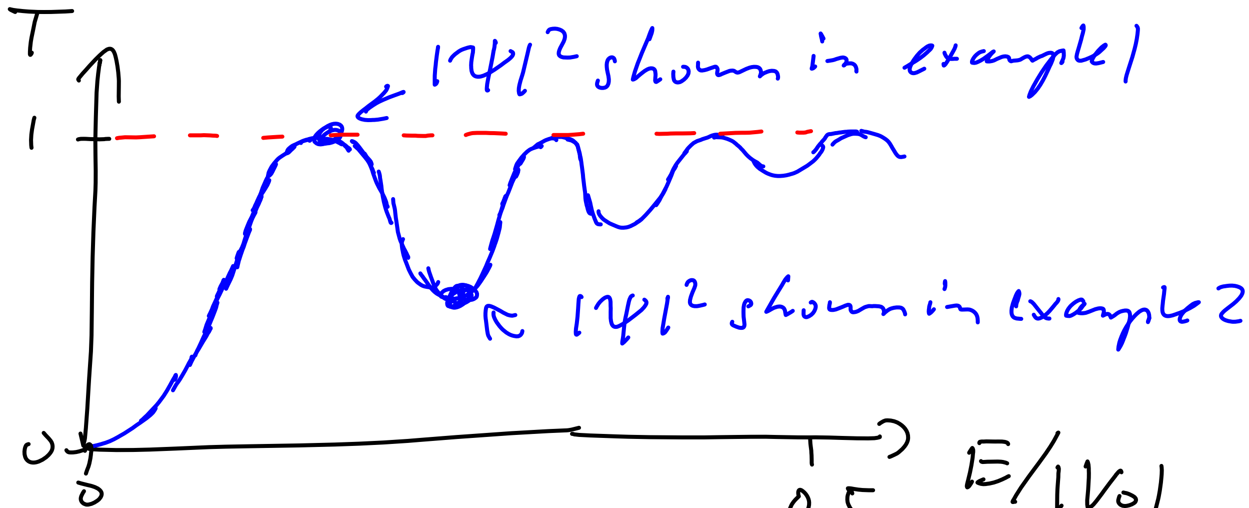
$$\Rightarrow T = \underset{\substack{\uparrow \\ \frac{k_1}{k_2} = 1}}{} \left| \frac{D}{A_0} \right|^2 = \left| \frac{4 k_1 k_2}{\pm\left[(k_2+k_1)^2 - (k_2-k_1)^2\right] e^{i k_1 L}} \right|^2$$

$$= \left| \frac{1}{\pm e^{i k_1 L}} \right|^2 = \underline{\underline{1}}$$

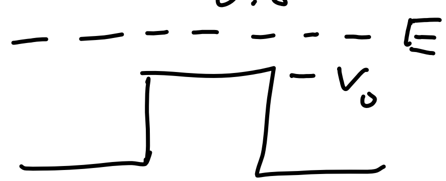
⇒ unity transmission at certain resonant particle energies ($R = 0$)

→ for well





→ similar for square barrier

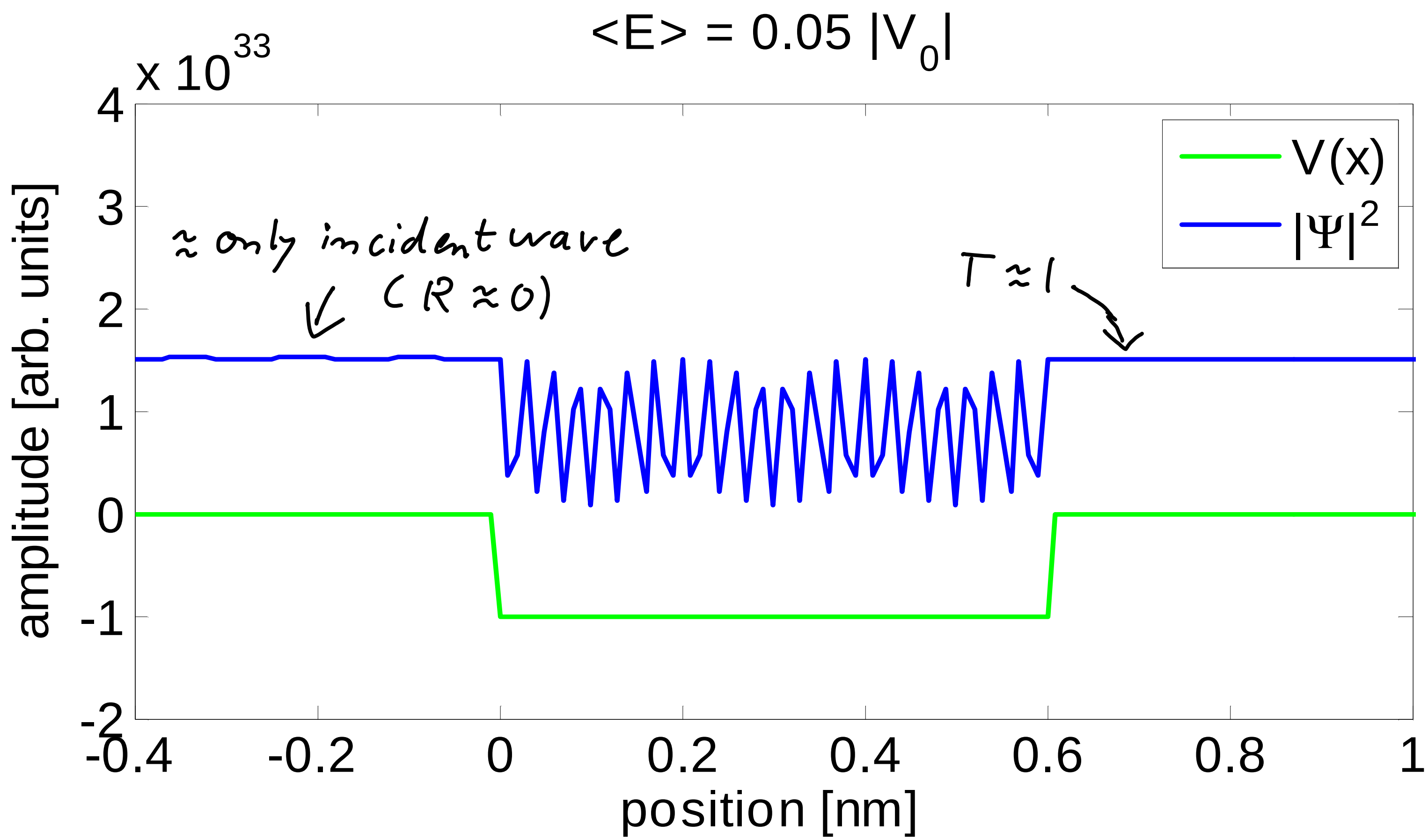


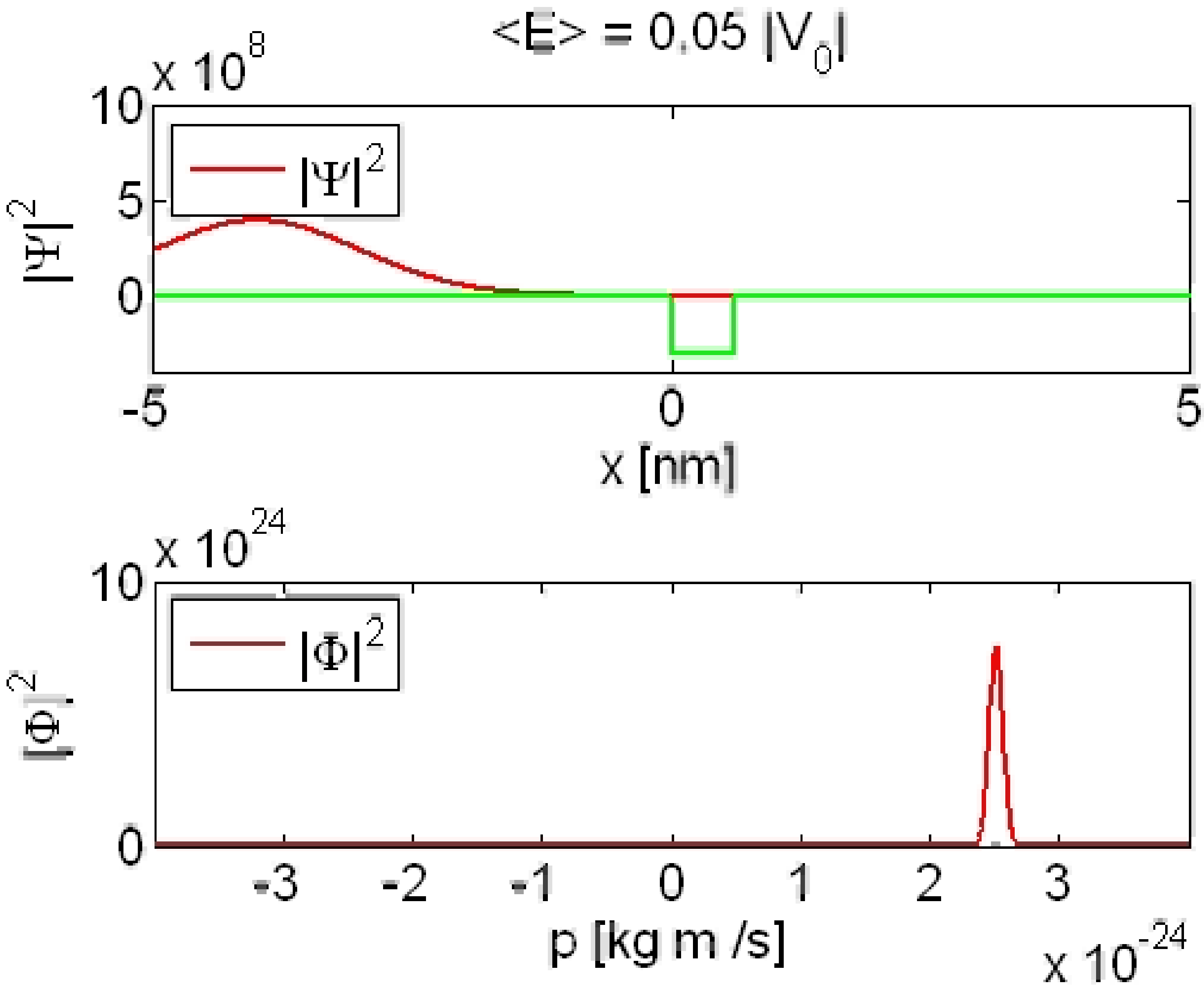
→ see homework

→ see example 3, 4

→ see Lab III

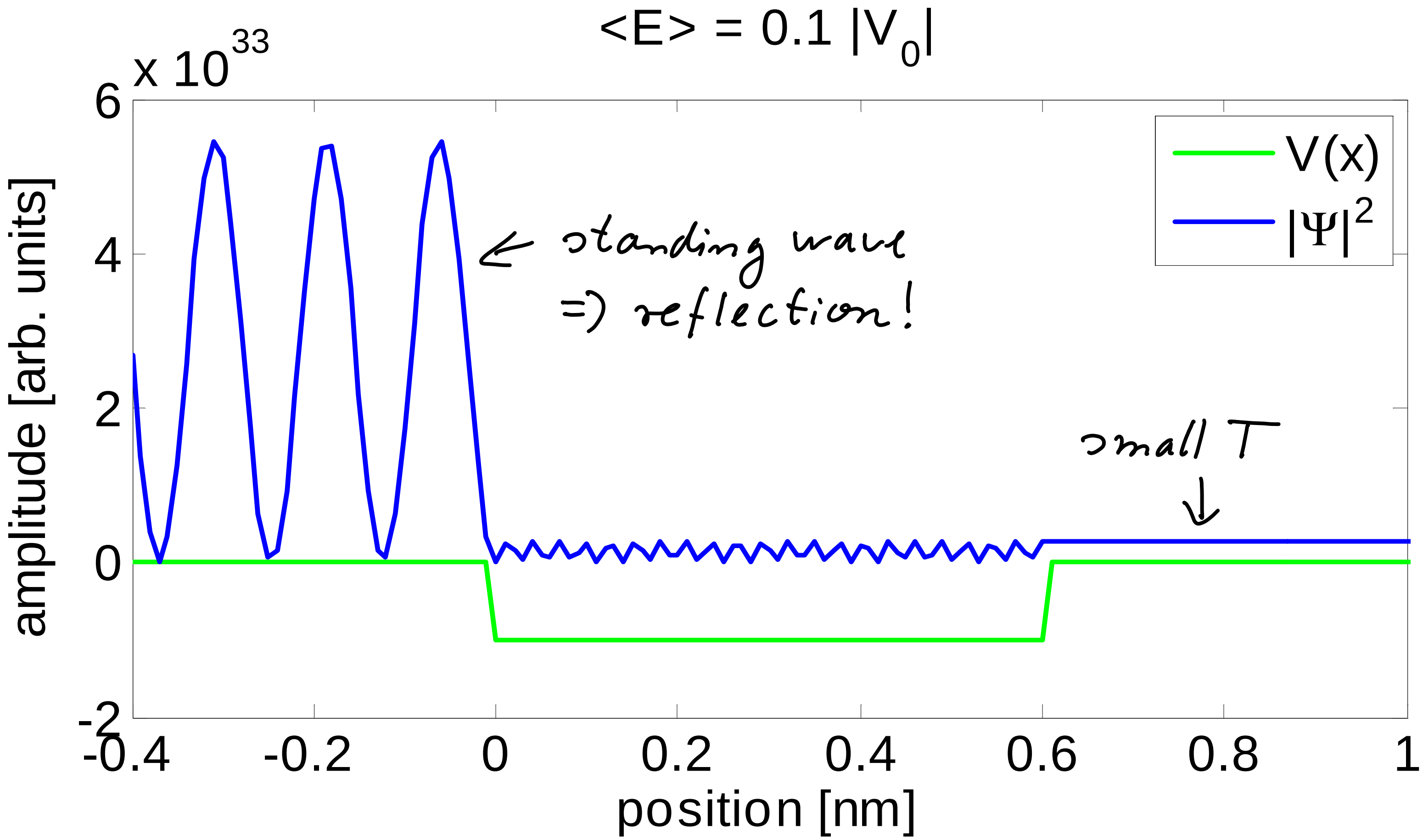
Example 1: Square well for $T \approx 1$ $(k_2 L = 22 \cdot \pi)$

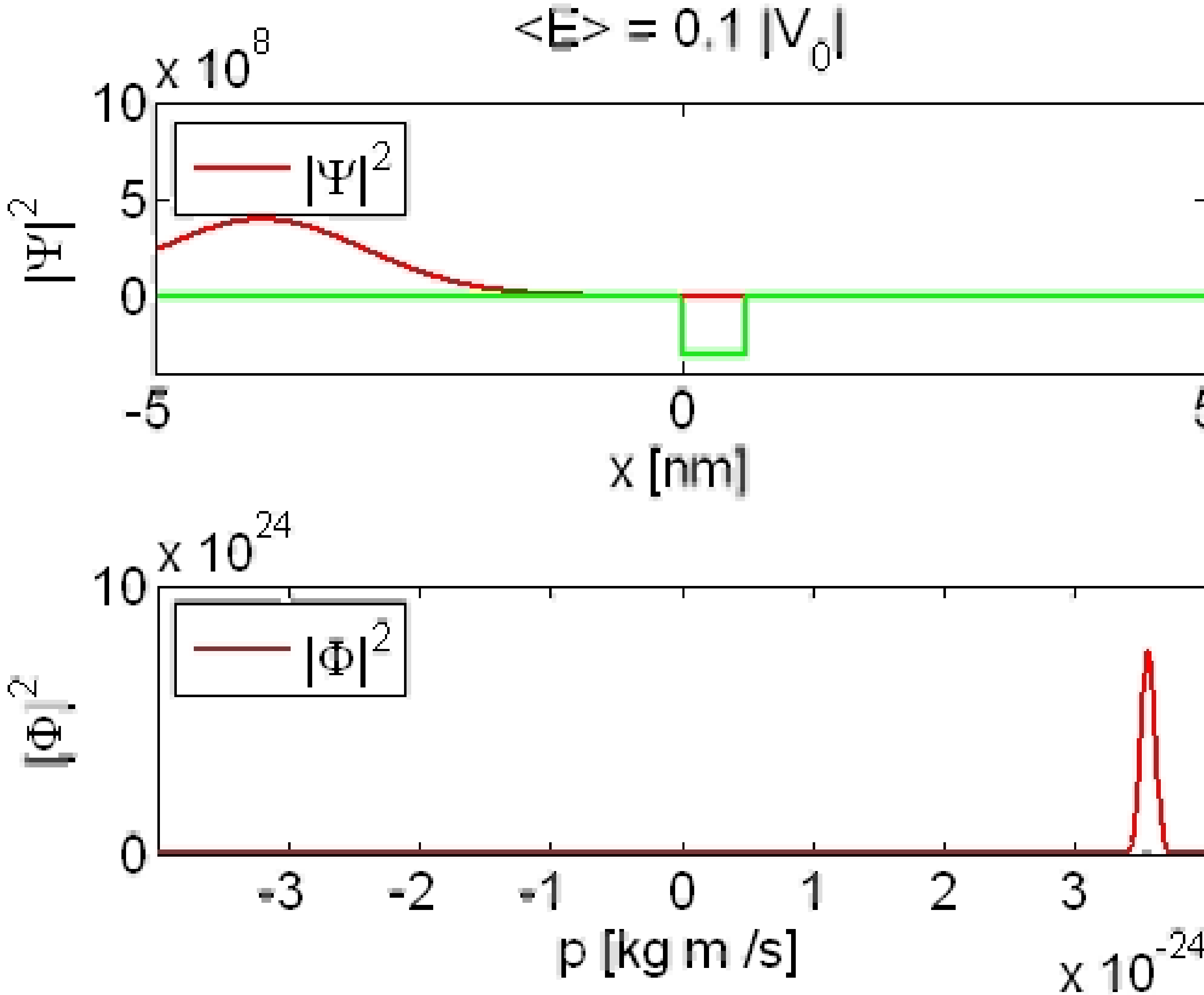




<E> = 0,05 |V0|
x 10^8
10
5
0
|Ψ|^2
-5
0
5
x [nm]
x 10^24
10
0
|Φ|^2
-3
-2
-1
0
1
2
3
p [kg m /s]
x 10^-24

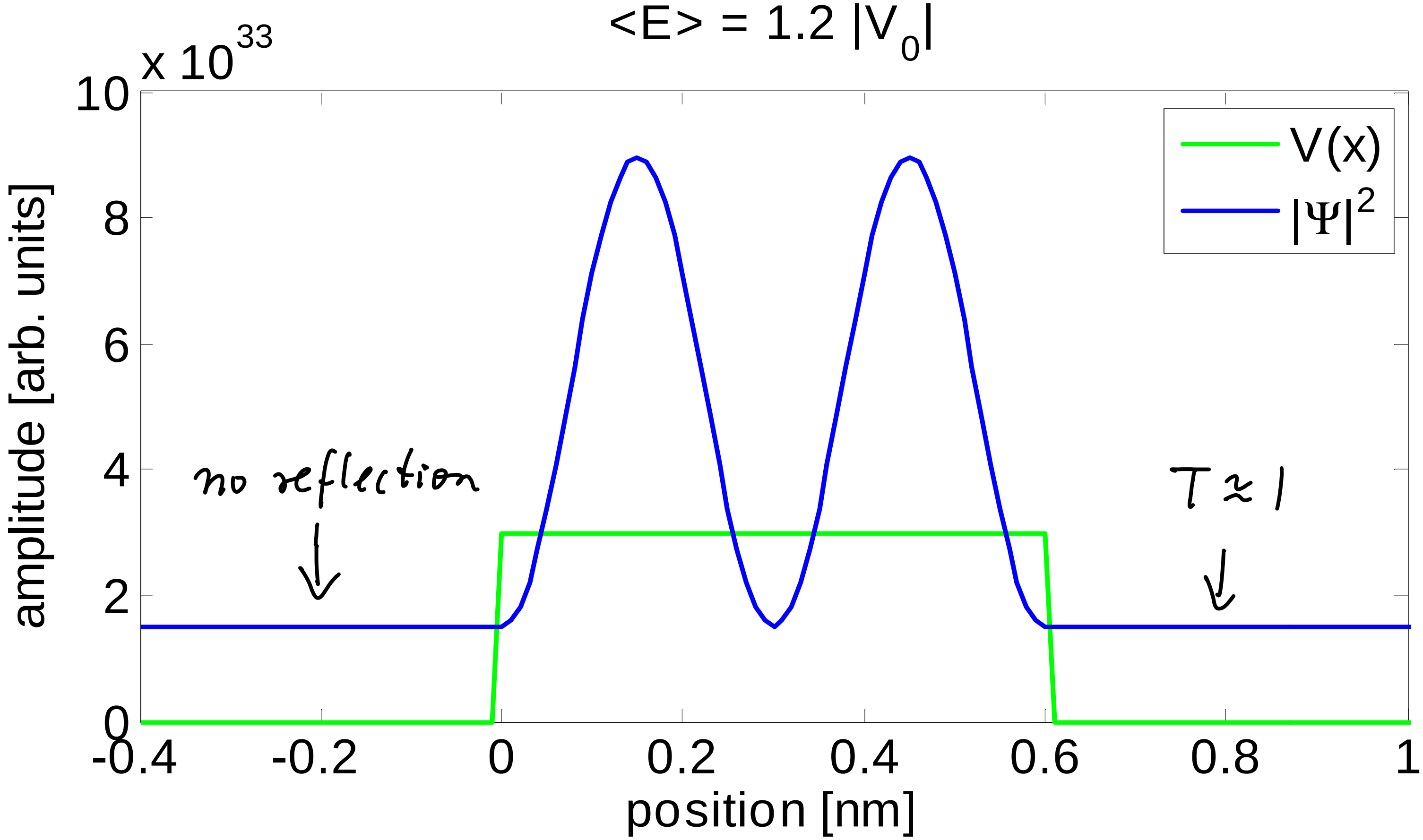
Example 2: *Square well* $(k_2 L \approx 22.5\,\pi)$

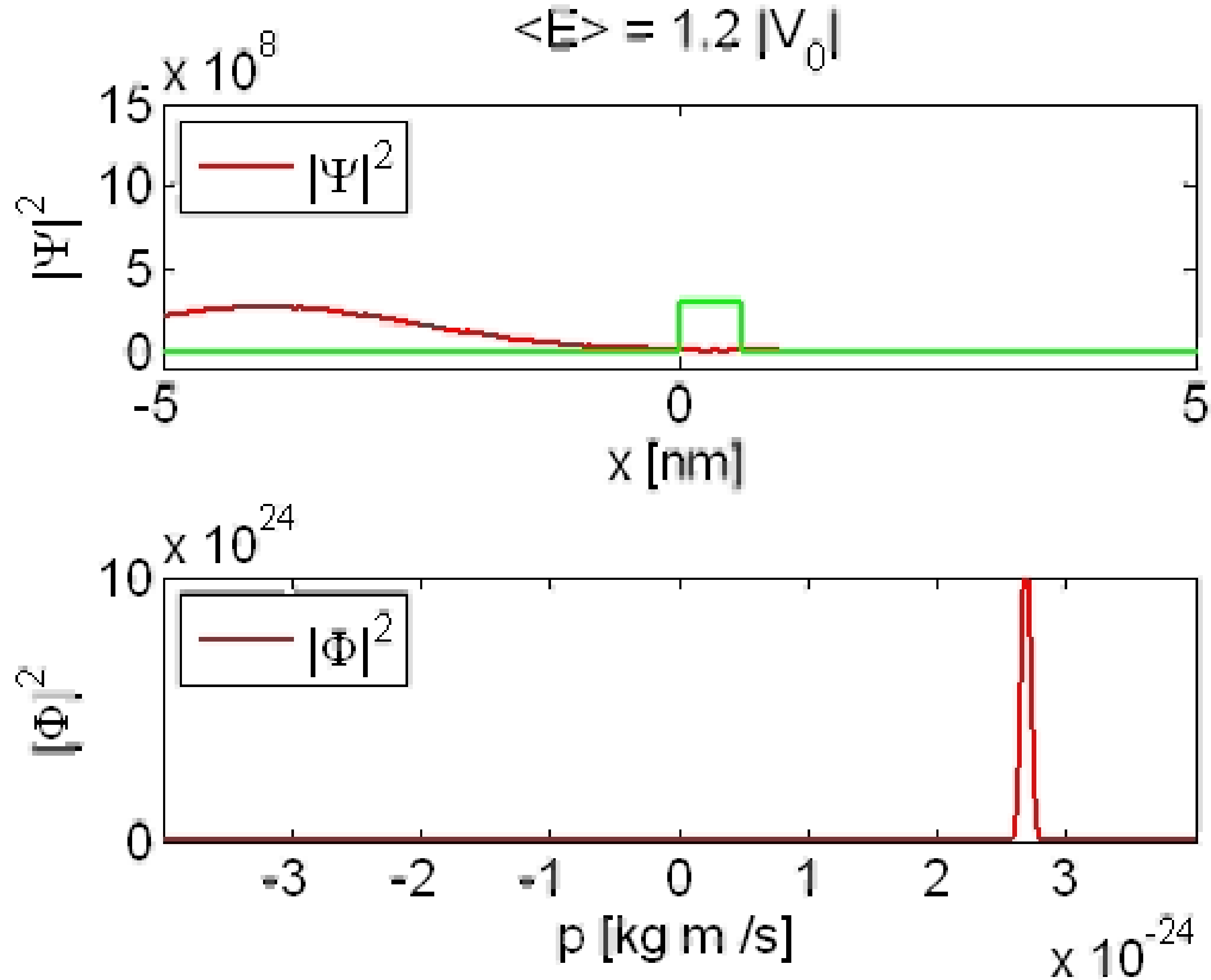




<E> = 0,1 |V0|
x 10^8
10
5
0
|Ψ|^2
-5
0
5
x [nm]
x 10^24
10
0
|Φ|^2
-3
-2
-1
0
1
2
3
p [kg m /s]
x 10^-24

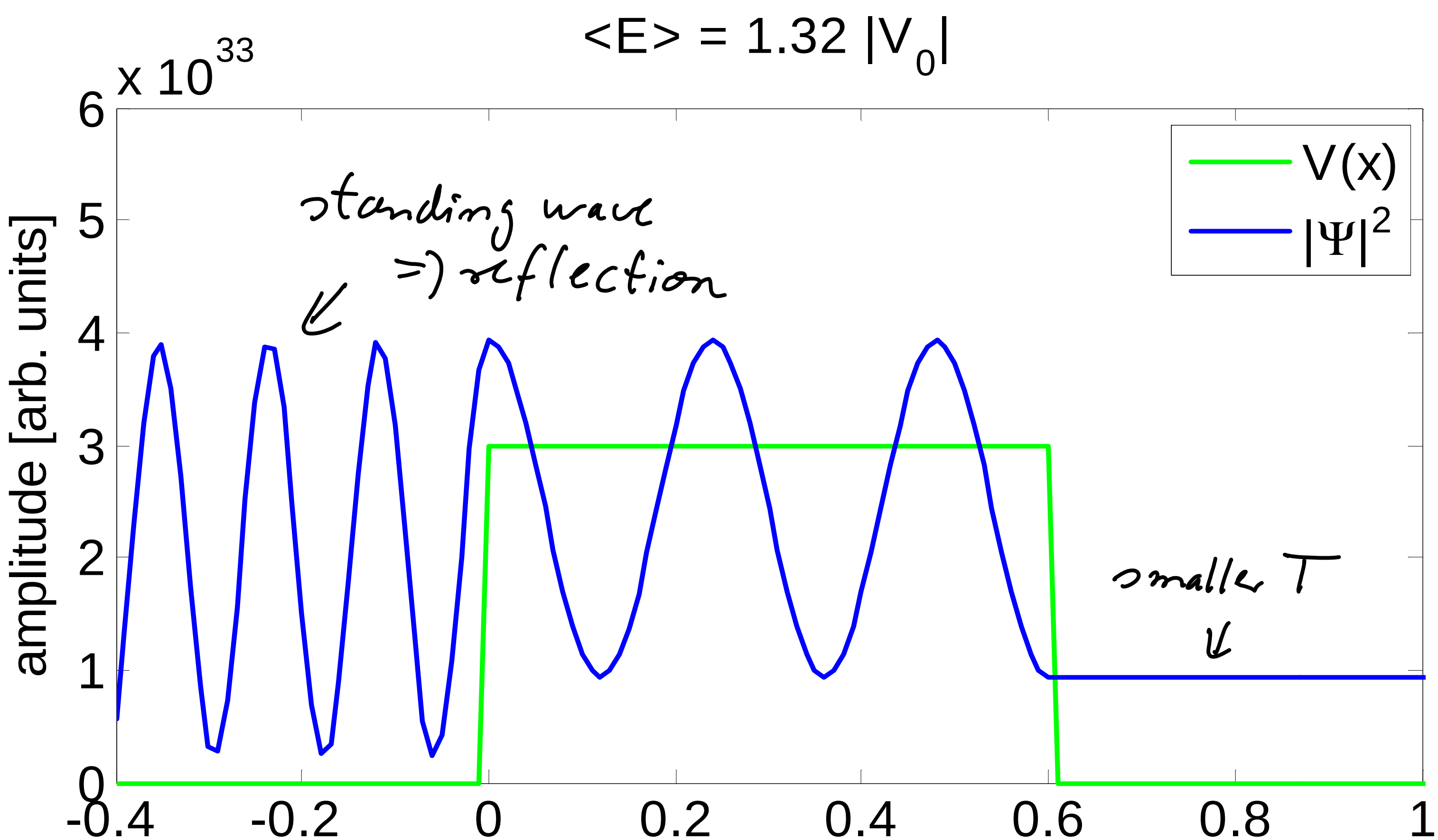
Example 3: Square barrier for $T \approx 1$ $(k_2 L = 2\pi)$

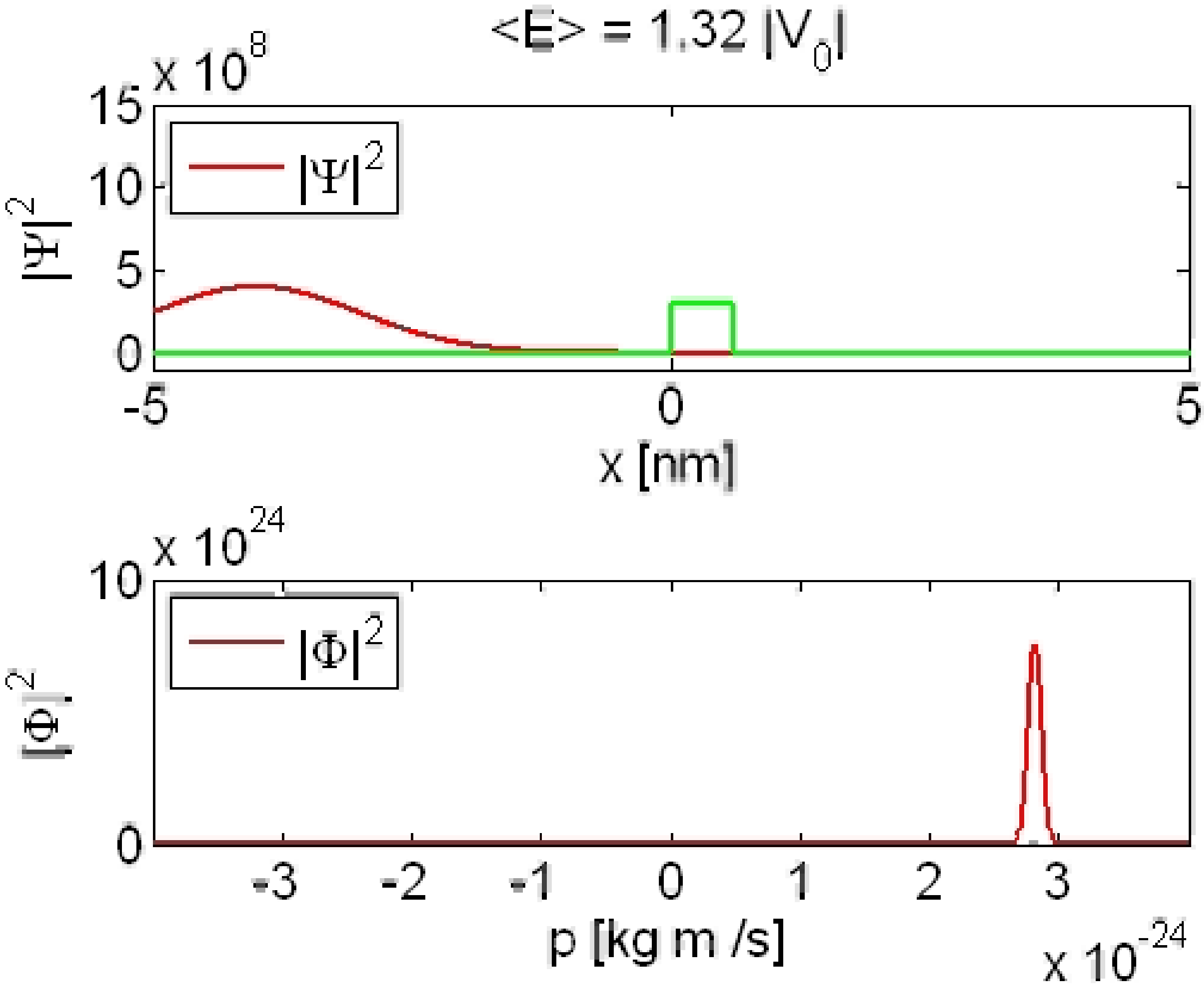




<E> = 1.2 |V0|
x 10^8
15
10
5
0
|Ψ|^2
-5
0
5
x [nm]
x 10^24
10
0
|Φ|^2
-3
-2
-1
0
1
2
3
p [kg m /s]
x 10^-24

Example 4: Square barrier ($\kappa_2 L = 2.5\pi$)





<E> = 1.32 $|V_0|$
x 10^8
15
10
5
0
$|\Psi|^2$
-5
0
5
x [nm]
x 10^{24}
10
0
$|\Phi|^2$
-3
-2
-1
0
1
2
3
p [kg m /s]
x 10^{-24}