Lecture 30: 04/06/09

- Generalized Uncertainty Principle
- Free particle II: particle dynamics

IV_3 The Free Particle at t=0:

Recap:

→ localized wave packet → $\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p,t)\, \underbrace{e^{i\frac{px}{\hbar}}}_{\text{state of definite momentum}}\, dp$

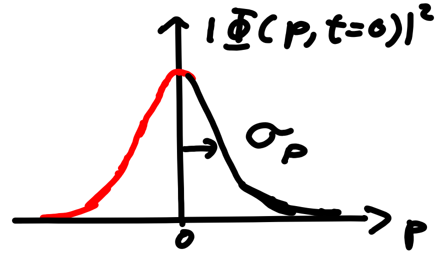
superposition ↗ ↑ quantum amplitude

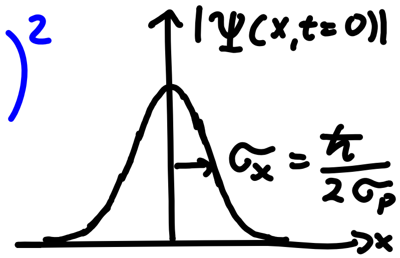
→ Example: gaussian wave packet:

$$\Phi(p,t=0) = \frac{1}{(2\pi)^{1/4}\sqrt{\sigma_p}}\, e^{-p^2/4\sigma_p^2}$$

⇕

$$\Psi(x,t=0) = \frac{1}{(2\pi)^{1/4}\sqrt{\frac{\hbar}{2\sigma_p}}}\, e^{-x^2/4\left(\frac{\hbar}{2\sigma_p}\right)^2}$$





$$\sigma_x\, \sigma_p = \frac{\hbar}{2}$$

IV_4 The Generalized Uncertainty Principle:

$$\boxed{\sigma_A^2\, \sigma_B^2 \underset{\text{always}}{\geq} \left(\frac{1}{2i}\langle[\hat{A},\hat{B}]\rangle\right)^2}$$

where : $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$: Commutator of $\hat{A}$ and $\hat{B}$

$$\sigma_A^2 \cdot \sigma_B^2 \geq \left(\frac{1}{2i} \left[\langle f|g\rangle - \langle g|f\rangle \right] \right)^2$$

$$\Downarrow \; ?$$

$$\sigma_A^2 \sigma_B^2 \underset{\text{always}}{\geq} \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

$$\langle f|g\rangle = \langle (\underbrace{\hat{A} - \langle A\rangle}_{\text{hermitian}})\Psi | (\hat{B} - \langle B\rangle)\Psi\rangle$$

$$= \langle \Psi | (\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\Psi\rangle$$

$$= \langle \Psi | \hat{A}\hat{B} - \hat{A}\langle B\rangle - \langle A\rangle \hat{B} + \langle A\rangle\langle B\rangle)\Psi\rangle$$

$$\Rightarrow \langle f|g\rangle = \langle \Psi|\hat{A}\hat{B}\Psi\rangle - \langle B\rangle\langle \Psi|\hat{A}\Psi\rangle - \langle A\rangle\langle \Psi|\hat{B}\Psi\rangle + \langle A\rangle\langle B\rangle\underbrace{\langle \Psi|\Psi\rangle}_{1}$$

$$= \langle \hat{A}\hat{B}\rangle - \langle B\rangle\langle A\rangle - \langle A\rangle\langle B\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle$$

→ Similar: (simply exchange A and B):

$$\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle B\rangle\langle A\rangle = \langle \hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle$$

$$\rightarrow \langle f|g\rangle - \langle g|f\rangle = \langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle$$

$$= \langle [\hat{A},\hat{B}]\rangle$$

where $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$: commutator of the two operators $\hat{A}$ and $\hat{B}$

generalized uncertainty principle

$$\boxed{\sigma_A^2 \sigma_B^2 \underset{\text{always}}{\geq} \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle\right)^2}$$

$\Rightarrow$ if operator $\hat{A}$ and $\hat{B}$ do not commute: (i.e $[\hat{A}, \hat{B}] \neq 0$)

→ uncertainty principle

→ incompatible observables

→ no shared complete set of eigenfunctions

→ on a single quantum system: sequence of measurements matters: different results! Measurements change state!

- **Example:**

suppose: $\hat{A} = x$ (position operator) $\quad \hat{B} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ (momentum operator)

$\Rightarrow$ what is $[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = ?$

$\rightarrow$ let this act on a test function $f(x)$

(never do math with operators without a test function!)

$$\underline{[\hat{x},\hat{p}]f(x)} = \left[x\frac{\hbar}{i}\frac{\partial f}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x}\left(xf(x)\right)\right]$$

$$= \frac{\hbar}{i}\left[x\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial x} - f(x)\right] = \underline{i\hbar f(x)}$$

$\Rightarrow [\hat{x},\hat{p}] = i\hbar \quad \Rightarrow \underline{\langle[\hat{x},\hat{p}]\rangle} = i\hbar\underbrace{\langle\Psi|\Psi\rangle}_{1} = \underline{i\hbar}$

$$\Rightarrow \sigma_x^2\sigma_p^2 \geq \left(\frac{1}{2i}i\hbar\right)^2 = \left(\frac{\hbar}{2}\right)^2 \Rightarrow \boxed{\sigma_x\sigma_p \geq \frac{\hbar}{2}}$$

Heisenberg's uncertainty principle!

• why inequality in $\sigma_x \sigma_p \geq \frac{\hbar}{2}$?

• compare:

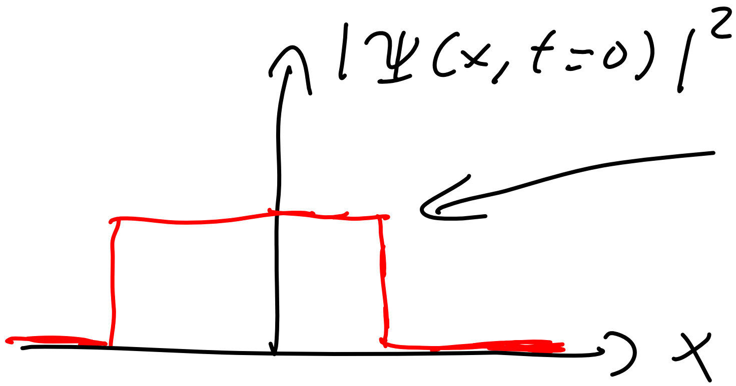
→

gaussian wave packet at $\underline{\underline{t=0}}$:

had $\sigma_x \sigma_p = \frac{\hbar}{2}$

minimum uncertainty

→



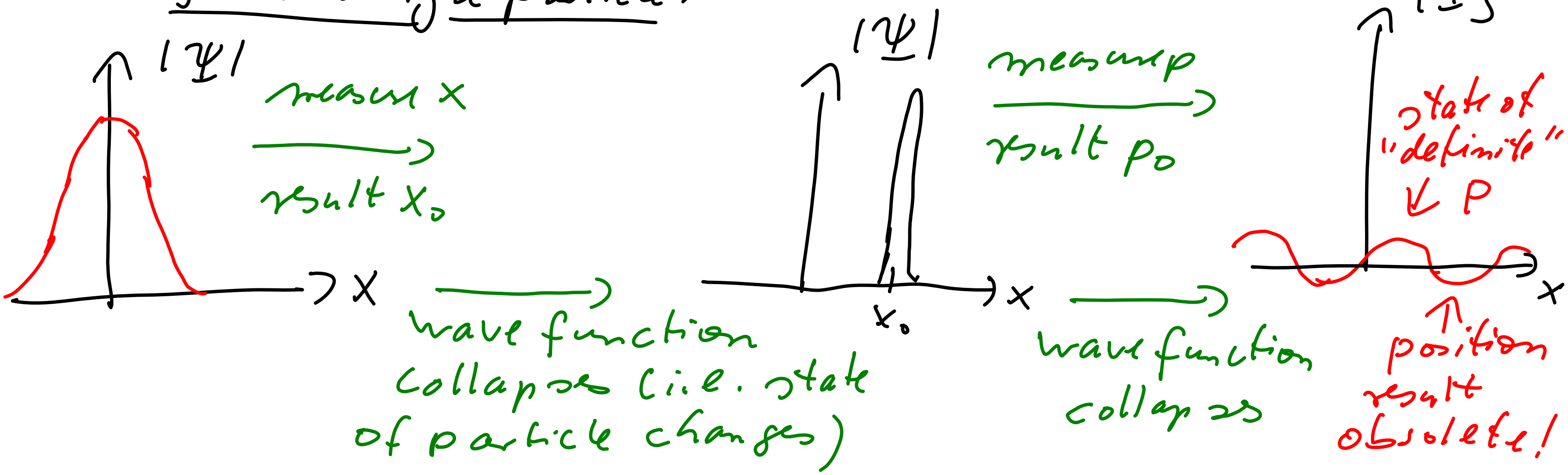
sharp corner!

⇒ will need lot's of Fourier components

⇒ large spread in p to get $\Psi(x,t=0)$

⇒ $\sigma_x \sigma_p \gg \frac{\hbar}{2}$ here

- **What does the Heisenberg Uncertainty Principle mean?**
  - Consequence of the statistical interpretation in quantum mechanics
  - need to consider <u>large</u> number of identically prepared quantum systems / particles!
    - $\rightarrow$ measure x on half and momentum p on <u>other</u> half $\Rightarrow$ calculate σ_x, σ_p from results
    - $\Rightarrow$ will always find $\sigma_x \sigma_p \geq \hbar/2$
  - <u>for a single particle</u>:



- **some numbers:**

$$\sigma_x \sigma_p \geq \frac{\hbar}{2} \Rightarrow \underset{\substack{\uparrow \\ \text{velocity} = p/m}}{\sigma_x \sigma_v} \geq \frac{\hbar}{2m} \simeq \begin{cases} 0.5 \ \frac{cm^2}{s} \text{ for } e^- \\ 5 \cdot 10^{-27} \ cm^2/s \\ \text{for } 0.1\,g \text{ pea} \end{cases}$$

example: for $\sigma_x = 0.5\,\mathring{A}$ (Bohr radius)

→ electron: $\sigma_v \geq 10^8\ cm/s$ large!

[classical velocity in Bohr atom $\simeq 10^9\ cm/s$]

→ pea:

$\sigma_v \geq 10^{-18}\ cm/s$ small!

⇒ Heisenberg's uncertainty principle does not matter for macroscopic particles!

- **Heisenberg uncertainty principle and ground state energies:**

for particle in ground state confined in well of some length scale σ_x:

3D potential well

kinetic energy: $\langle KE \rangle = \dfrac{\langle p^2 \rangle}{2m} \overset{\downarrow}{=} \dfrac{\langle p_x^2 \rangle + \langle p_y^2 \rangle + \langle p_z^2 \rangle}{2m}$

$$= \frac{3\,\sigma_{p_x}^2}{2m} \geq \frac{3}{2m}\left(\frac{\hbar}{2}\frac{1}{\sigma_x}\right)^2$$

for 3D and $\nearrow$ $\qquad\qquad$ $\uparrow \sigma_x \sigma_{p_x} \geq \hbar/2$

$\langle p_x^2 \rangle = \langle p_y^2 \rangle = \langle p_z^2 \rangle$

(symmetric)

$$\sigma_{p_x} = \sqrt{\langle p_x^2 \rangle - \underbrace{\langle p_x \rangle^2}_{\text{Ohere}}}$$

$$\Rightarrow \sigma_{p_x} = \sigma_{p_y} = \sigma_{p_z}$$

Example: proton in nucleus

$m_p = 1.7 \cdot 10^{-27}\,kg \qquad \sigma_x \approx 10^{-14}\,m$

$$\Rightarrow \langle KE \rangle \approx \frac{3}{2 \cdot 1.7\, 10^{-27}\,kg}\left(\frac{10^{-34}\,Js}{2 \cdot 10^{-14}\,m}\right)^2$$

$\approx 0.1\ MeV \Leftarrow$ correct order magnitude for nuclear binding energies!

IV$_5$ The Free Particle II: Particle Dynamics:

so far: wave packet at $t=0$: $\Psi(x,t=0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p,t=0)\, e^{i\frac{px}{\hbar}}\, dp$

$\Rightarrow$ How do they envolve in time?

$\Rightarrow$ Need to include time-dependence of the state of definite momentum and energy (for free particle!)

$\Rightarrow e^{-iE/\hbar \cdot t}$

$$\Rightarrow \Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p)\, e^{ipx/\hbar}\, e^{-i\frac{E(p)}{\hbar}t}\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p)\, e^{ipx/\hbar}\, e^{-i\frac{p^2}{2m}\frac{t}{\hbar}}\, dp \quad \leftarrow E = \frac{p^2}{2m} \text{ for } \underline{\underline{\text{free}}} \text{ particle}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p,t)\, e^{ipx/\hbar}\, dp$$

with the momentum space wave function

$$\Phi(p,t) = \Phi(p)\, e^{-i\, p^2/2m \; t/\hbar}$$ for the free particle

Note: $\Rightarrow |\Phi(p,t)|^2$ is time-indep. for the free particle

free particle $\leftrightarrow$ no forces $\leftrightarrow$ $\vec{F} = \frac{d\vec{p}}{dt} \overset{!}{=} 0$