Lecture 29: 04/03/09

- The free particle I (at $t=0$)
- Heisenbergs Uncertainty principle

Recap

IV_2 Expect. Values in Position and Momentum Space:

operator	position space	momentum space
position op. $\hat{x}$	x	$-\frac{\hbar}{i}\frac{\partial}{\partial p}$
momentum op. $\hat{p}$	$\frac{\hbar}{i}\frac{\partial}{\partial x}$	p

$$\Rightarrow \langle Q(x,p)\rangle = \int_{-\infty}^{+\infty} \Psi^* \hat{Q}\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi\, dx = \int_{-\infty}^{+\infty} \Phi^*(p,t)\, \hat{Q}\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p\right)\Phi(p,t)\, dp$$

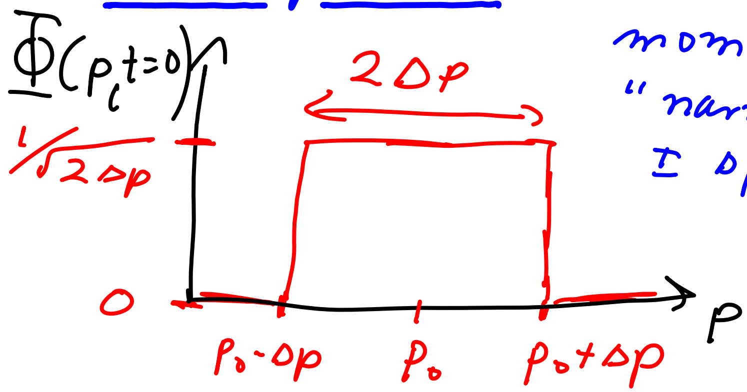
IV_3 The Free Particle:

$\rightarrow$ localized wave packet $\longrightarrow$ $$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p,t)\, \underbrace{e^{i\frac{px}{\hbar}}}_{\text{state of definite momentum}}\, dp$$

superposition

quantum amplitude

- **Example I:**



momentum in "narrow range" Δp about p_0

$$\Phi(p, t=0) = \begin{cases} \dfrac{1}{\sqrt{2\Delta p}} & \text{for } p_0 - \Delta p \le p \le p_0 + \Delta p \\ 0 & \text{elsewhere} \end{cases}$$

normalized!

$\Rightarrow$ wave packet in position space:

$$\Psi(x, t=0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{p_0 - \Delta p}^{p_0 + \Delta p} \underbrace{\frac{1}{\sqrt{2\Delta p}}}_{\Phi} e^{ipx/\hbar}\, dp$$

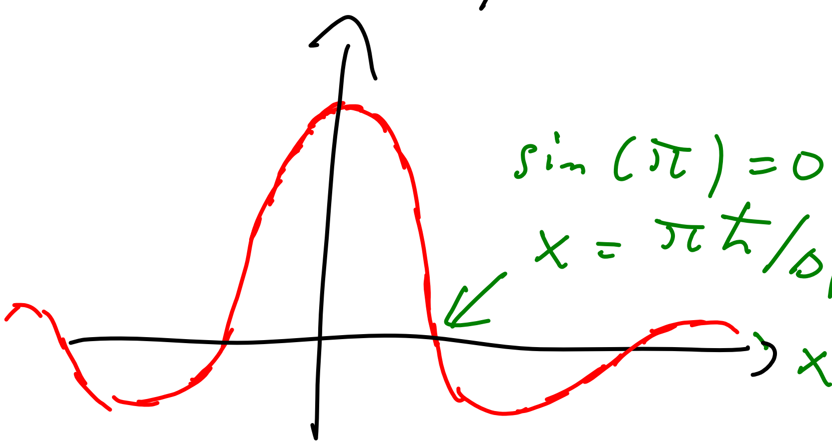
$$= \frac{1}{2\sqrt{\pi\hbar\Delta p}} \int_{p_0 - \Delta p}^{p_0 + \Delta p} e^{ipx/\hbar}\, dp = \frac{1}{2\sqrt{\pi\hbar\Delta p}} \frac{\hbar}{ix} e^{ipx/\hbar} \Bigg|_{p_0 - \Delta p}^{p_0 + \Delta p}$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{\pi\Delta p}} \frac{1}{ix} \left\{ e^{i\frac{x}{\hbar}(p_0 + \Delta p)} - e^{i\frac{x}{\hbar}(p_0 - \Delta p)} \right\}$$

$$\Rightarrow \Psi(x,t=0) = \frac{1}{2}\sqrt{\frac{\hbar}{\pi\Delta p}}\,\frac{1}{ix}\,e^{i\frac{x}{\hbar}p_0}\Big\{\cos\left(\frac{x}{\hbar}\Delta p\right)+i\sin\left(\frac{x}{\hbar}\Delta p\right)$$
$$-\cos\left(\frac{x}{\hbar}\Delta p\right)+i\sin\left(\frac{x}{\hbar}\Delta p\right)\Big\}$$

$$\Rightarrow \Psi(x,t=0) = \sqrt{\frac{\hbar}{\pi\Delta p}}\;\underbrace{e^{i\frac{x}{\hbar}p_0}}_{\substack{\text{(traveling)}\\ \text{wave with}\\ \text{momentum } p_0}}\;\underbrace{\frac{\sin\left(\frac{x\Delta p}{\hbar}\right)}{x}}_{\substack{\text{envelope}\\ \text{function from}\\ \text{"spread" } 2\Delta p\\ \text{in momentum}}}$$

$\sin\left(\frac{x\Delta p}{\hbar}\right)/x$



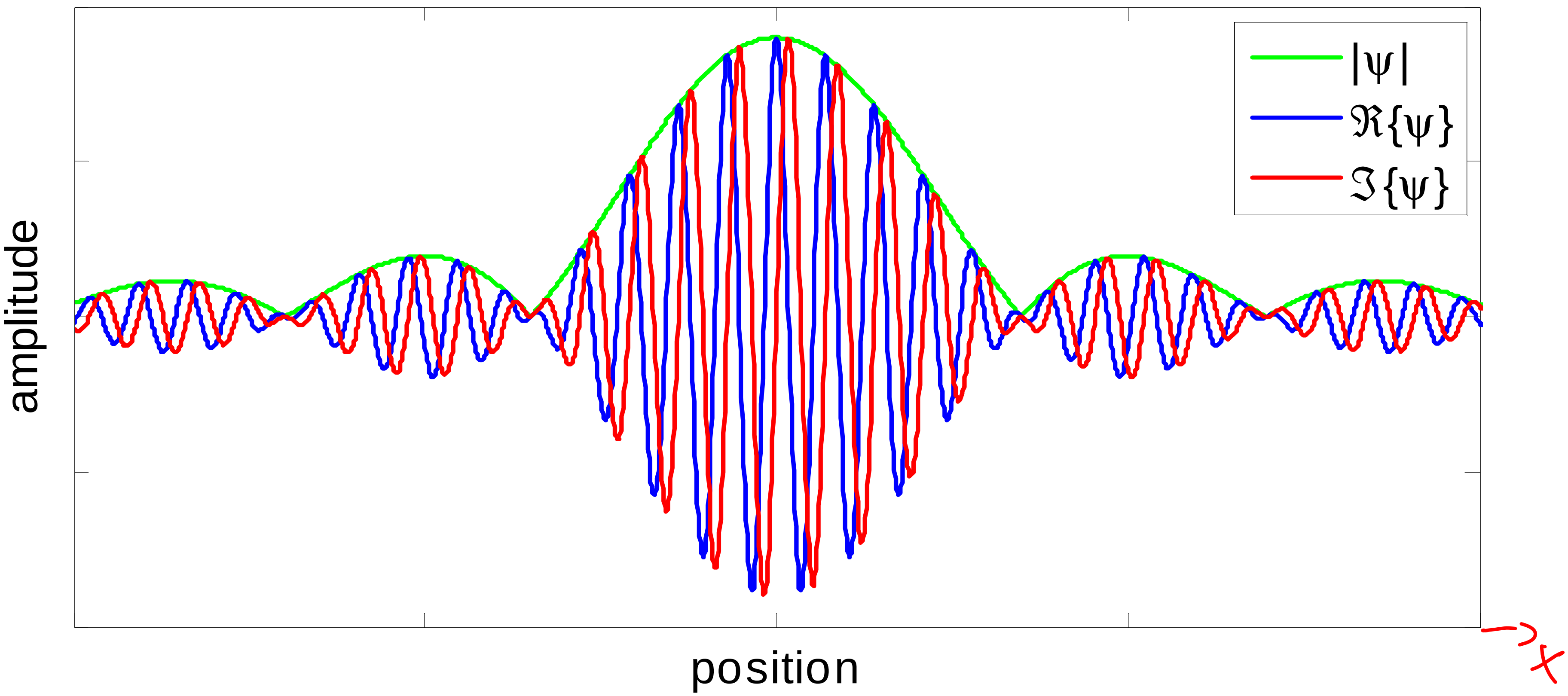
$\sin(\pi)=0$

$x = \pi\hbar/\Delta p \equiv \Delta x \quad \Rightarrow \quad \underline{\Delta x\cdot\Delta p \simeq \pi\hbar}$ (here)

related quantities!

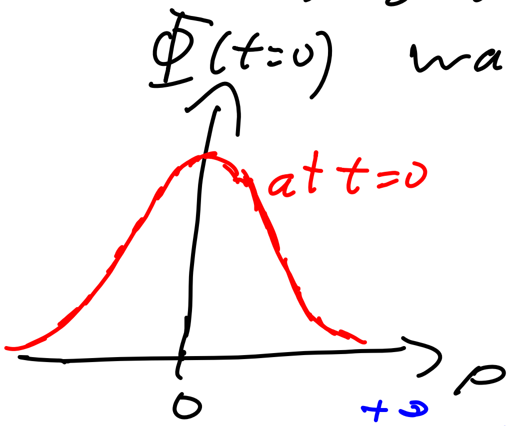
$\Rightarrow$ large Δp gives small Δx and vica versa!

$\Rightarrow$ see Heisenberg's uncertainty principle for example I: $(t=0)$



- **Example II: Gaussian wave packet:** at $t=0$

→ start with Gaussian momentum space wave function:



$$\Phi(p, t=0) = \underbrace{\frac{1}{(2\pi)^{1/4}\sqrt{\sigma_p}}}_{\text{normalized!}} e^{-p^2/(2\sigma_p)^2}$$

$$\Rightarrow \langle p \rangle = \int_{-\infty}^{+\infty} \Phi^*(p,t=0)\, p\, \Phi(p,t=0)\, dp = 0 \quad (\text{odd function!})$$

$$\underline{\underline{\langle p^2 \rangle}} = \int_{-\infty}^{+\infty} \Phi^*(p,t=0)\, p^2\, \Phi(p,t=0)\, dp$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_p} \int_{-\infty}^{+\infty} p^2 e^{-2p^2/4\sigma_p^2}\, dp \Leftarrow \int_{-\infty}^{+\infty} x^2 e^{-x^2/a^2} = \sqrt{\pi}\,\frac{a^3}{2}$$

$a = \sqrt{2}\,\sigma_p$ ⇓

$$= \frac{1}{\sqrt{2\pi}\,\sigma_p} \sqrt{\pi}\, \frac{(\sqrt{2}\,\sigma_p)^3}{2} = \underline{\underline{\sigma_p^2}}$$

⇒ standard deviation (rms fluctuation of measured momentum about average $\langle p \rangle = 0$ on large # of identically prepared particles)

$$\underline{\underline{\sigma_p}} = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \underline{\underline{\sigma_p}}$$

→ calculate corresponding position space wavefunction

$$\Psi(x,t=0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p,t=0)\, e^{ipx/\hbar}\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar\sigma_p}} \frac{1}{(2\pi)^{1/4}} \int_{-\infty}^{+\infty} \underbrace{e^{-p^2/4\sigma_p^2}\, e^{ipx/\hbar}}_{\text{define: } y \equiv \frac{1}{2\sigma_p}\left[p - i\frac{2x\sigma_p^2}{\hbar}\right]}\, dp$$

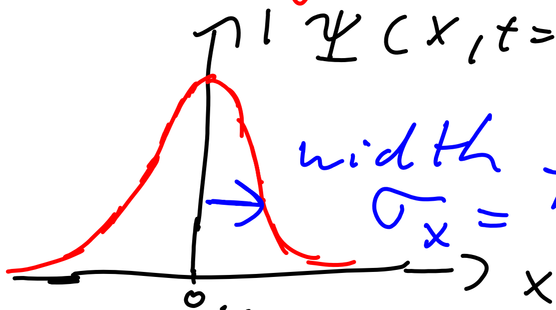
$$= \frac{2\sigma_p}{\sqrt{2\pi\hbar\sigma_p}\,(2\pi)^{1/4}} \underbrace{\int_{-\infty}^{+\infty} e^{-y^2}\, e^{-x^2\sigma_p^2/\hbar^2}\, dy}_{\text{use } \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}}$$

result:

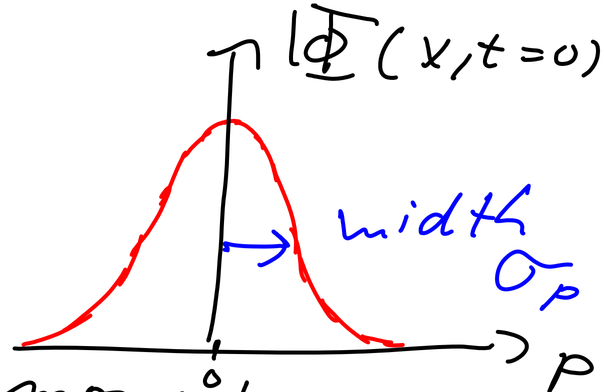
$$\Psi(x,t=0) = \frac{1}{(2\pi)^{1/4}\sqrt{\frac{\hbar}{2\sigma_p}}}\, e^{-x^2 \Big/ 4\underbrace{\left(\frac{\hbar}{2\sigma_p}\right)^2}_{\sigma_x^{\,2}}}$$

(compare eq. for Φ and Ψ)

$\Rightarrow$ Fourier transformation of a gaussian is a gaussian!



$\longleftrightarrow$



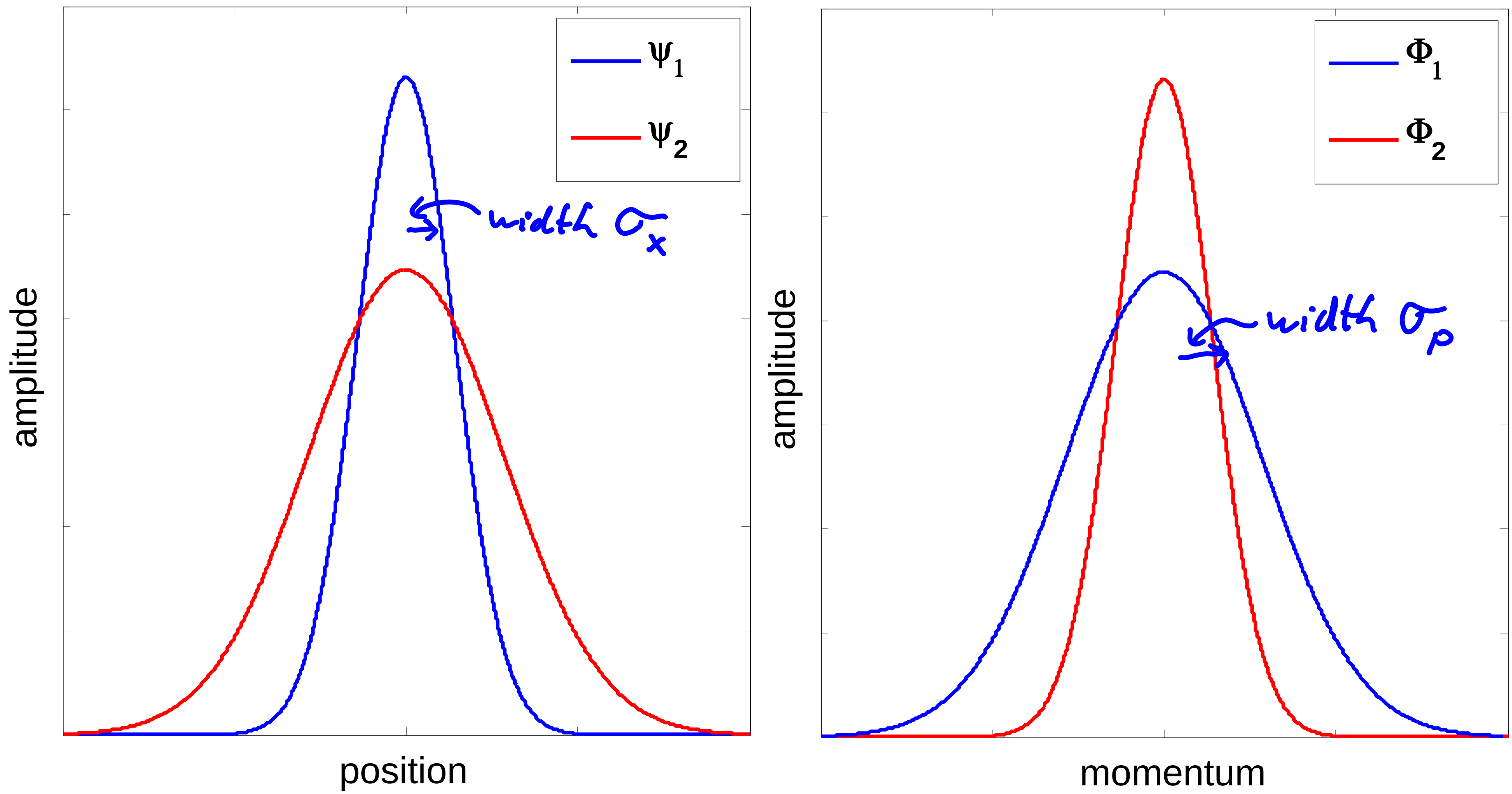
$$\Rightarrow \langle x \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t=0)\, x\, \Psi(x,t=0)\, dx = 0$$

$$\langle x^2 \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t=0)\, x^2\, \Psi(x,t=0)\, dx = \left(\frac{\hbar}{2\sigma_p}\right)^2$$

$\Rightarrow$ standard deviation $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \dfrac{\hbar}{2\sigma_p}$

for gaussian wave packet: $\Rightarrow$ $\sigma_x \cdot \sigma_p = \hbar/2$

→ Gaussian wave packets:



$\sigma_x \cdot \sigma_p = \hbar/2$ here

IV_4 The Generalized Uncertainty Principle:

Saw: $\sigma_x \sigma_p \geq$ (always) some minimum value $= \left(\frac{\hbar}{2}\right)$

(Heisenberg's Uncertainty Principle)

gaussian is case with minimum uncertainty

Proof: Generalized Uncertainty Principle!

→ any observable A:

$$\text{variance} = \sigma_A^2 = \langle \hat{A}^2 \rangle - \langle A \rangle^2 = \langle (\hat{A} - \langle A \rangle)^2 \rangle$$

$$= \langle \Psi | (\hat{A} - \langle A \rangle)^2 | \Psi \rangle$$

$$= \langle (\hat{A} - \langle A \rangle)\Psi \mid (\hat{A} - \langle A \rangle)\Psi \rangle = \langle f | f \rangle$$

$\hat{A} - \langle A \rangle$ is a hermitian operator

with $f \equiv (\hat{A} - \langle x \rangle)\Psi$

→ any observable B: $\sigma_B^2 = \langle g | g \rangle$ with $g \equiv (\hat{B} - \langle B \rangle)\Psi$

$\rightarrow$ $\underline{\sigma_A^2 \sigma_B^2} = \langle f|f\rangle\langle g|g\rangle \geq \underline{|\langle f|g\rangle|^2}$

$\uparrow$ Schwartz inequality

[recall vectors: $\underbrace{|\vec{f}|^2}_{\vec{f}\cdot\vec{f}} \cdot |\vec{g}|^2 \geq |\vec{f}\cdot\vec{g}|^2 = |\vec{f}|^2 \cdot |\vec{g}| \cdot \cos^2\varphi$

$\rightarrow$ for <u>any</u> complex number:

$$|z|^2 = \{Re(z)\}^2 + \{Im(z)\}^2 \geq \{Im(z)\}^2$$

$$= \left[\frac{1}{2i}(z - z^*)\right]^2$$

$\rightarrow$ let $z = \langle f|g\rangle$

$\langle f|g\rangle^* = \langle g|f\rangle$

$$\Rightarrow \sigma_A^2 \cdot \sigma_B^2 \geq \left(\frac{1}{2i}\left[\langle f|g\rangle - \langle g|f\rangle\right]\right)^2$$

$$\Rightarrow \sigma_A^2 \sigma_B^2 \underset{\text{always!}}{\geq} \left(\frac{1}{2i}\langle \underbrace{[\hat{A}, \hat{B}]} \rangle\right)^2$$

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$$