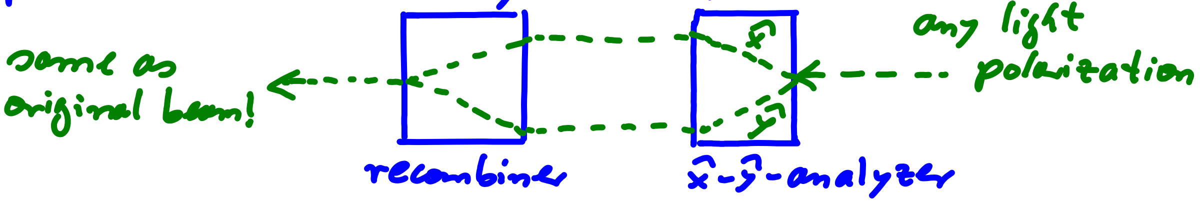
Lecture 27: 03/30/09

- Momentum space

② Linear polarization analyzer loop: Recap

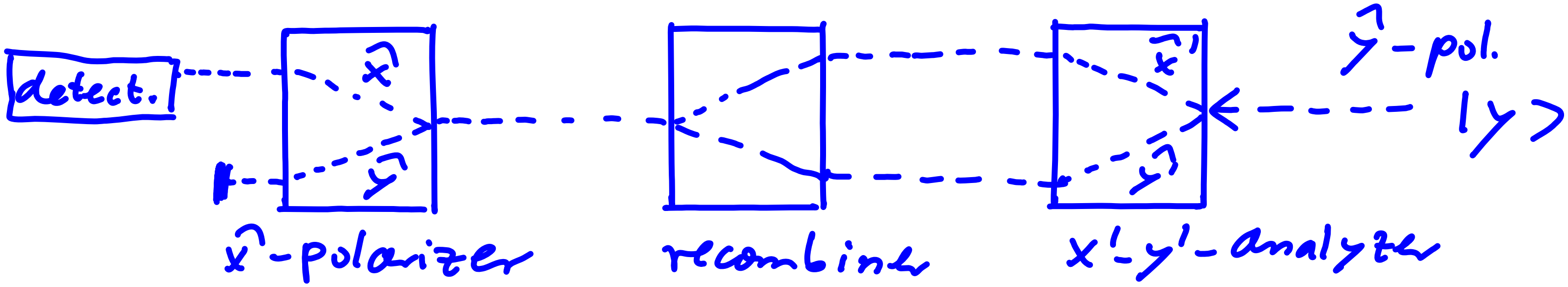


-> Classically: $\vec{E} = \hat{x}(\hat{x}\cdot\vec{E}) + \hat{y}(\hat{y}\cdot\vec{E})$

-> QM: $|\psi\rangle = |x\rangle\langle x|\psi\rangle + |y\rangle\langle y|\psi\rangle$

$= |x'\rangle\langle x'|\psi\rangle + |y'\rangle\langle y'|\psi\rangle$ } change of basis

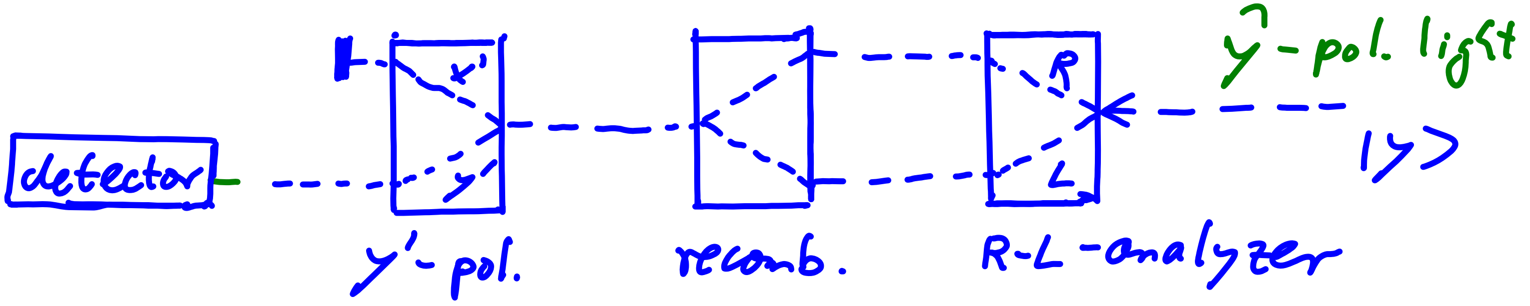
③ Example of Interference of Polarization States:

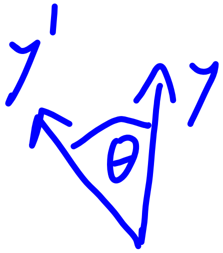


total projection ampl. $= \langle x|y'\rangle\langle y'|y\rangle + \langle x|x'\rangle\langle x'|y\rangle$

$$= \langle x|\{\underbrace{|y'\rangle\langle y'| + |x'\rangle\langle x'|}_{=\text{unity operator!}}\}|y\rangle = \langle x|y\rangle = 0$$

④ Circular polarized light / complex projection amplitudes





$$\langle y'|y\rangle = \cos\theta = \langle y'|\underbrace{\{|R\rangle\langle R| + |L\rangle\langle L|\}}_{\substack{\text{RL analyzer-loop} \\ \text{(unity operator)}}}|y\rangle$$

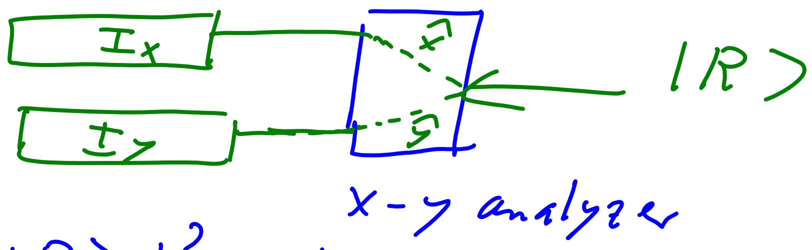
$$= \underbrace{\langle y'|R\rangle\langle R|y\rangle}_{\text{path I}} + \underbrace{\langle y'|L\rangle\langle L|y\rangle}_{\text{path II}}$$

$\Rightarrow$ what are $\langle R|y\rangle$, $\langle L|y\rangle$, ... ?

→ classically: circular pol. light has equal components in $\hat{x}$ and $\hat{y}$ direction, only out of phase

equal! $\nearrow$ $I_x \propto |\langle x|R\rangle|^2$

$\searrow$ $I_y \propto |\langle y|R\rangle|^2$



$\Rightarrow |\langle y|R\rangle|^2 = |\langle x|R\rangle|^2 = \frac{1}{2}$

$\Rightarrow$ same for left circ. pol.:

$$|\langle y|L\rangle|^2 = |\langle x|L\rangle|^2 = \frac{1}{2}$$

$\Rightarrow$ try with real projection ampl.:

$$\left(\pm\frac{1}{\sqrt{2}}\right)\left(\pm\frac{1}{\sqrt{2}}\right) + \left(\pm\frac{1}{\sqrt{2}}\right)\left(\pm\frac{1}{\sqrt{2}}\right) \overset{?}{=} \cos\theta$$

$\Rightarrow$ does not work! $\Rightarrow$ need complex proj. ampl. to include angle θ between $\hat{y}$ and $\hat{y}'$

$\rightarrow$ try: $$\left.\begin{aligned} |L\rangle &= \frac{1}{\sqrt{2}}\left\{ i\,|x\rangle + |y\rangle \right\} \\ |R\rangle &= \frac{1}{\sqrt{2}}\left\{ (-i)\,|x\rangle + |y\rangle \right\} \end{aligned}\right\} \text{see lecture } \underline{24}$$

$$\Rightarrow \langle y'|R\rangle\langle R|y\rangle + \langle y'|L\rangle\langle L|y\rangle$$

$$= \underbrace{\langle y'|R\rangle}_{?}\langle y|R\rangle^* + \langle y'|L\rangle\langle y|L\rangle^* = \langle y'|R\rangle\frac{1}{\sqrt{2}} + \langle y'|L\rangle\frac{1}{\sqrt{2}}$$

$$|y'\rangle = |x\rangle\langle x|y'\rangle + |y\rangle\langle y|y'\rangle$$

$$= |x\rangle\cos\left(\theta + \frac{\pi}{2}\right) + |y\rangle\cos\theta = |x\rangle(-\sin\theta) + |y\rangle\cos\theta$$

$\Rightarrow$ bra: $\langle y'| = \langle x|y'\rangle^*\langle x| + \langle y|y'\rangle^*\langle y| = (-\sin\theta)\langle x| + \cos\theta\,\langle y|$

$$\Rightarrow \langle y'|R\rangle\langle R|y\rangle + \langle y'|L\rangle\langle L|y\rangle =$$

$$= \left\{(-\sin\theta)\frac{1}{\sqrt{2}}(-i) + \cos\theta\,\frac{1}{\sqrt{2}}\right\}\frac{1}{\sqrt{2}} + \left\{(-\sin\theta)\frac{1}{\sqrt{2}}\,i + \cos\theta\,\frac{1}{\sqrt{2}}\right\}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\left\{\cos\theta + i\sin\theta\right\} + \frac{1}{2}\left\{\cos\theta - i\sin\theta\right\} = \underline{\underline{\cos\theta}}$$

$\Rightarrow$ works out ... $\Rightarrow$ projection ampl. can be complex!

IV Momentum Space, Wave packets and the Heisenberg Uncertainty Principle

IV$_1$ Position and Momentum Space:

	position space	momentum space
$|\Psi\rangle$ in terms of basis states	$|\Psi\rangle = \int |x\rangle\langle x|\Psi\rangle dx$	$|\Psi\rangle = \int |p\rangle\langle p|\Psi\rangle dp$
basis states:	$|x\rangle$: states of definite position are solutions of $\underset{\uparrow}{\hat{x}}|x=y\rangle = \overset{\text{value}\,\swarrow}{y}|x=y\rangle$ position operator	$|p\rangle$ states of definite momentum are solutions of $\underset{\uparrow}{\hat{p}}|p\rangle = \overset{\text{eigen value}\,\swarrow}{p}|p\rangle$ momentum operator
basis states are "orthonormal"	$\langle x|x'\rangle = \underset{\uparrow}{\delta}(x-x')$ dirac delta function	$\langle p|p'\rangle = \delta(p-p')$

	position space	momentum space
projection amplitudes	$\langle x \mid \Psi\rangle = \Psi(x,t)$ position space wave function $\Rightarrow \lvert\Psi\rangle = \int \lvert x\rangle \Psi(x,t)\,dx$ $\lvert\langle x \mid \Psi\rangle\rvert^2 dx = \lvert\Psi(x,t)\rvert^2 dx$ = probability of finding particle between x and x+dx	$\langle p \mid \Psi\rangle = c_p \equiv \Phi(p,t)$ momentum space wave function $\lvert\Psi\rangle = \int \lvert p\rangle \Phi(p,t)\,dp$ $\lvert\langle p \mid \Psi\rangle\rvert^2 dp = \lvert\Phi(p,t)\rvert^2 dp$ = prob that measurement of momentum will give result within p and p+dp
Normalization	$\int_{-\infty}^{+\infty} \lvert\Psi(x,t)\rvert^2 dx = 1$	$\int_{-\infty}^{+\infty} \lvert\Phi(p,t)\rvert^2 dp = 1$

- How to get $\Psi(x,t)$ from $\Phi(p,t)$ and vica versa?

$\Rightarrow$ change of basis stats!

$$\Psi(x,t) = \underset{\substack{\uparrow \\ \text{projection onto} \\ \text{position space}}}{\langle x|\Psi\rangle} = \langle x| \underbrace{\left\{ \int_{-\infty}^{+\infty} |p\rangle\langle p|\Psi\rangle \, dp \right\}}_{|\Psi\rangle}$$

$$\Rightarrow \Psi(x,t) = \int_{-\infty}^{+\infty} \langle x|p\rangle \underbrace{\langle p|\Psi\rangle}_{\Phi(p,t)} \, dp = \int_{-\infty}^{+\infty} \Phi(p,t) \underline{\underline{\overbrace{\langle x|p\rangle}^{\text{proj. of one basis state onto another}}}} \, dp$$

$\Phi(p,t)$: momentum space wave function

similiar:

$$\Phi(p,t) = \langle p|\Psi\rangle = \langle p| \left\{ \int_{-\infty}^{+\infty} |x\rangle\langle x|\Psi\rangle \, dx \right\}$$

$$= \int_{-\infty}^{+\infty} \langle p|x\rangle \underbrace{\langle x|\Psi\rangle}_{=\Psi(x,t)} \, dx = \int_{-\infty}^{+\infty} \Psi(x,t) \underline{\underline{\langle p|x\rangle}} \, dx$$

need: $\langle x | p \rangle = \langle p | x \rangle^*$: projection ampl. of state of definite momentum onto state of def. position

recall:

for arbitary state: $\langle x | \Psi \rangle = \Psi(x)$

$$\langle x | p \rangle = \underbrace{f_p(x)}_{\text{position space wave function for (non-physical) particle of definite momentum}}$$

$\Rightarrow$ $f_p(x)$ is the eigenfunction of the momentum operator:

$$\hat{p}\, f_p(x) = \frac{\hbar}{i} \frac{\partial}{\partial x} f_p(x) = p\, f_p(x)$$ (see coop)

$$\Rightarrow \quad \boxed{f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}\, e^{i p x/\hbar} = \langle x | p \rangle}$$

so that $\langle f_p | f_{p'} \rangle = \delta(p - p')$

→ Result:

position space wave function:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Phi(p,t)\, e^{ipx/\hbar}\, dp$$

momentum space wave function:

from $\langle p|x\rangle = \langle x|p\rangle^*$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \Psi(x,t)\, e^{-ipx/\hbar}\, dx$$

Note: - Both types of wave function contain the same information!

→ can calculate expectation values from Ψ or Φ

- If $\Psi(x,t)$ is normalized ⟷ $\Phi(p,t)$ will be normalized too!

↑ above equations (not for equ. in F&T!)

- $\Phi(p,t)$ is essentially the Fourier-transformation of $\Psi(x,t)$, and $\Psi(x,t)$ is the inverse Fourier-transformation of $\Phi(p,t)$

→ Side note: Fourier-transformation

– $F(k)$ is the Fourier-transform of $f(x)$:

$$\boxed{F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx}$$

$$k = \frac{2\pi}{\lambda}$$

– $f(x)$ is the inverse Fourier-transform of $F(k)$:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k)\, e^{+ikx}\, dk}$$

⇒ set $p = \hbar k$, $F = \Phi$, $\Psi(x,t) = f(x) \cdot \sqrt{\hbar}$

to get equations for $\Psi(x,t)$ and $\Phi(p,t)$

from previous page...