**Lecture 24: 03/23/08**

- Formalism III
  - Dirac notation
  - A two basis states system
- Photon Polarization States

Recap

## $III_4$ Eigenfunctions of a hermitian operator:

$$\boxed{\hat{Q} f(x) = q f(x)}$$ are

- orthonormal: $\langle f_n | f_m \rangle = \delta_{nm}$
- complete

⇒ can expand any wave function in terms of the (base) functions:

different base/ set of eigenf.

$$\boxed{\Psi(x,t) = \sum_n c_n(t) f_n(x)} = \sum_m c_m'(t) f_m'(x)$$

with $\boxed{c_n(t) = \langle f_n | \Psi(x,t) \rangle}$

$$\Psi(x,t) = \int c(q,t) f_q(x)\, dq$$ with $c(q,t) = \langle f_q | \Psi \rangle$

Recap

## $III_5$ Generalized Statistical Interpretation:

- If one measures an observable $Q$ on a particle in the state $\Psi(x,t)$, one is certain to get one of the eigenvalues of the hermitian operator $\hat{Q}$.
- Probability of getting eigenvalue $q_n$ associated with eigenfunction $f_n(x)$ is $|c_n|^2 = |\langle f_n | \Psi \rangle|^2$
- Upon measurement, the wave function "collapses" to the corresponding eigenstate!

Expectation values:

$$\boxed{\langle Q \rangle = \sum_n q_n |c_n|^2}$$

# III$_6$ Dirac Notation:

Dirac: chop bracket notation for inner product in two pieces:

bra: $\langle \alpha |$ $\leftrightarrow$ instruction to integrate / row vector

ket: $| \beta \rangle$ $\leftrightarrow$ state vector

- **ket $|\beta\rangle$:**

"all that can be known"

- represents the state of the system/particle
- in position space: $|\beta\rangle$ is represented by a function $\Psi(x,t)$
- in vector space: state $|\beta\rangle$ is represented by a state vector:

$$|\beta\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- Example: system with only two linear independent states:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Operator $\hat{Q}$:

transforms one ket into another:

$$\hat{Q}|\beta\rangle = |\beta'\rangle$$

• **bra $\langle\alpha|$:**

- in position space: instruction to integrate

$$\langle f| = \int_{-\infty}^{+\infty} f^* [\underbrace{\ldots\ldots}_{\text{function from ket}}] \, dx$$

- in vector space:

bra is row vector $\langle\alpha| = (a_1^*, a_2^* \ldots a_n^*)$

$\Rightarrow$ if $\{|e_n\rangle\}$ is a complete set of discrete, orthonormal basis states: $\langle e_m | e_n \rangle = \delta_{mn}$

$\Rightarrow$ can expand any particle state $|\alpha\rangle$ in terms of $\{|e_n\rangle\}$

$$|\alpha\rangle = \sum_n c_n |e_n\rangle$$

with $c_n = \langle e_n | \alpha \rangle$ = projection amplitude, quantum amplitude

$\uparrow$ complex number

$$\Rightarrow |\alpha\rangle = \sum_n \underbrace{|e_n\rangle\langle e_n|}_{\text{call } \hat{P}}\alpha\rangle = \sum_n |e_n'\rangle\langle e_n'|\alpha\rangle$$

$$= \Big(\sum_n |e_n\rangle\langle e_n|\Big)|\alpha\rangle$$

$\hat{P} = |e_n\rangle\langle e_n|$ projection operator

projects state $|\alpha\rangle$ onto basis state $|e_n\rangle$

for vectors: $\vec{a} = \sum_n \hat{n}\,(\hat{n}\cdot\vec{a})$

$\uparrow$ unit vector

• Unity operator:

$$\sum_n |e_n\rangle\langle e_n| = 1$$ (if it acts on $|\alpha\rangle$, one gets $|\alpha\rangle$)

• $$\sum_n |\langle e_n|\alpha\rangle|^2 = \left\{\begin{array}{l}\text{sum of all amplitude-} \\ \text{square projection} \\ \text{amplitudes}\end{array}\right\} = \sum |C_n|^2 = 1$$

• Schrödinger Equation:

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle$$

# III$_7$ Example: A 2-state system

**(crude model for neutrino oscillation)**

two linearly indep. states : $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ general state: $|S\rangle = a|1\rangle + b|2\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

with $|a|^2 + |b|^2 = 1$

$\Rightarrow$ Schrödinger's equation (time dep.)

$$i\hbar \frac{\partial}{\partial t} |S\rangle = H|S\rangle$$

consider specific Hamiltonian:

$$H = \begin{pmatrix} f & g \\ g & f \end{pmatrix}$$

$\Rightarrow$ given at $t=0$: $|S(t=0)\rangle = |1\rangle \Rightarrow |S(t)\rangle = ?$

1) Solve time-indep. S.E.:

$$H\,|e_n\rangle = E_n\,|e_n\rangle$$

↑ state of determinate energy

$\Rightarrow$ eigenvalue $E_n$

characteristic equation: $\det\begin{pmatrix} f-E & g \\ g & f-E \end{pmatrix} \stackrel{!}{=} 0 \Rightarrow E_\pm = f \pm g$

$\Rightarrow$ normalized eigenvectors:

$$|e_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

2) Expand initial state as a linear combination of the eigenstates of $H$:

$$|S(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left\{ |e_+\rangle + |e_-\rangle \right\}$$

3) get $|S(t)\rangle$

→ "tack on" standart time dependence for states of determinate energy (stationary states)

$$e^{-i\frac{E_n}{\hbar}t}$$

$$\Rightarrow |S(t)\rangle = \frac{1}{\sqrt{2}}\left\{e^{-i(f+g)t/\hbar}|e_+\rangle + e^{-i(f-g)t/\hbar}|e_-\rangle\right\}$$

$$= e^{-ift/\hbar}\begin{pmatrix}\cos(gt/\hbar)\\ -i\sin(gt/\hbar)\end{pmatrix}$$

⇒ if $g \neq 0$: oscillats between $|1\rangle$ and $|2\rangle$

⇒ for neutrinos: : $|1\rangle$ : electron neutrino

$|2\rangle$ : muon neutrino

| $\hat{H}$ for $\infty$ square well | position operator | momentum operator |
|---|---|---|
| $\hat{H}\psi = E\psi$ | $x\, g_y(x) = y\, g_y(x)$ | $\frac{\hbar}{i}\frac{d}{dx} f_p(x) = p\, f_p(x)$ |
| $\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$ | $g_y(x) = \delta(x-y)$ | $f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ |
| $\langle \psi_n \mid \psi_m \rangle = \delta_{nm}$ | $\langle g_{y'} \mid g_y \rangle = \delta(y'-y)$ | $\langle f_{p'} \mid f_p \rangle = \delta(p'-p)$ |
| $\Psi(x,t) = \sum_n c_n(t)\,\psi_n(x)$ | $\Psi(x,t) = \int c(y,t)\, g_y(x)\, dy$ | $\Psi(x,t) = \int c(p,t)\, f_p(x)\, dp$ |
| $c_n(t) = \langle \psi_n \mid \Psi(x,t) \rangle$<br>$= \int_{-\infty}^{+\infty} \psi_n^* \Psi(x,t)\, dx$ | $c(y,t) = \langle g_y \mid \Psi(x,t) \rangle$<br>$= \int_{-\infty}^{+\infty} \delta(x-y) \Psi(x,t)\, dx = \Psi(y,t)$ | $c(p,t) = \langle f_p \mid \Psi(x,t) \rangle$<br>$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx$ |
| $\lvert c_n \rvert^2$ = prob. of measur. $E_n$ | $\lvert c(y) \rvert^2 dy$ = prop. of meas. position in range $(y, y+dy)$ | $\lvert c(p) \rvert^2 dp$ = prob. of meas. momentum in range $(p, p+dp)$ |
| $c_n(t) = \langle n \mid \Psi \rangle$<br>$n^{th}$ state<br>$\hat{H} \lvert n \rangle = E \lvert n \rangle$ | $\Psi(x,t) = \langle x \mid \Psi \rangle$<br>eigenfunct. of $\hat{x}$ with eigen-value $x$<br>$\hat{x} \lvert x \rangle = y \lvert x \rangle$ | $c(p) = \langle p \mid \Psi \rangle$<br>eigenfunct. of $\hat{p}$ with eigen-value $p$<br>$\hat{p} \lvert p \rangle = p \lvert p \rangle$ |
| $\lvert \Psi \rangle = \sum_n \lvert n \rangle \langle n \mid \Psi \rangle$ | $\lvert \Psi \rangle = \int \lvert x \rangle \langle x \mid \Psi \rangle\, dx$ | $\lvert \Psi \rangle = \int \lvert p \rangle \langle p \mid \Psi \rangle\, dp$ |

# $III_8$ Another example: Photon Polarization States

- Points to be shown:
  - orthogonality and completeness of basis states
  - projection amplitudes $\langle e_n | \alpha \rangle$
  - expansion of any possible polarization state in terms of basis states:

$$|\alpha\rangle = \sum_n |e_n\rangle\langle e_n|\alpha\rangle$$

  - can express same state $|\alpha\rangle$ in terms of different basis states
  - measurements change state; probabilities

- Math note: $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$z = a + ib \rightarrow \mathrm{Re}\{z\} = a \quad \mathrm{Im}(z) = b$

- Light

| Classically | QM picture |
|---|---|
| electromagnetic wave | photons |
| state of plane wave is uniquely specified by: | state of photon is uniquely specified by: |
| – frequency | – frequency / energy: $E_p = h\nu$ |
| – amplitude of electric field | – direction of motion |
| – direction of motion | – polarization |
| – polarization | |
| Intensity $\propto \overline{\lvert E\rvert^2}$ | Intensity $\propto \dfrac{\#\text{ of photons} \cdot E_p}{\Delta t}$ |

In the following: all waves / photons have same frequency and direction of motion (+ z-direction)

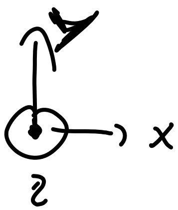
# Polarization of Light:

| classically | photon |
| --- | --- |

• light with arbitrary polarization:

prop. in +z direction

$$\vec{E}(z,t) = \mathrm{Re}\left\{ \left( a\, e^{i\varphi_a}\, \hat{x} + b\, e^{i\varphi_b}\, \hat{y} \right) e^{i(kz-\omega t)} \right\}$$




$$= a \cos(kz-\omega t+\varphi_a)\,\hat{x} + b\cos(kz-\omega t+\varphi_b)\,\hat{y}$$

amplitude $\geq 0$ relative phase unit vector in y-direct.

: superposition of two linear indep. waves

$|\Psi\rangle$

• Special cases: Classically | photon

1) linear polarized in x-direction:

$\Rightarrow b = 0 \qquad \longleftrightarrow \vec{E}$

$|x\rangle$

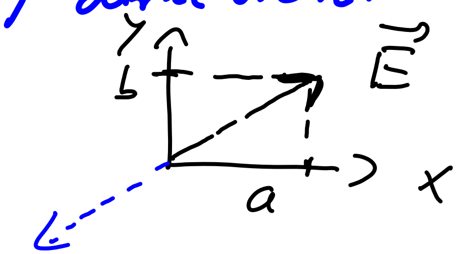
2) linear polarized in $\hat{y}$-direction

$\Rightarrow a = 0 \qquad \updownarrow \vec{E}$

$|y\rangle$

3) linear polarized in $a\hat{x} + b\hat{y}$ direction:

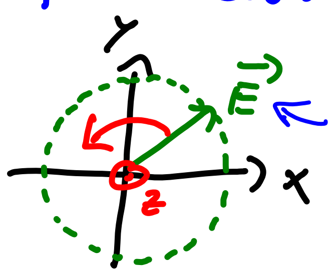
$\Rightarrow a, b \neq 0 \; , \; \varphi_a = \varphi_b$




$\frac{1}{\sqrt{a^2+b^2}} \{ a|x\rangle + b|y\rangle \}$

normalize!

4) left circular polarized:



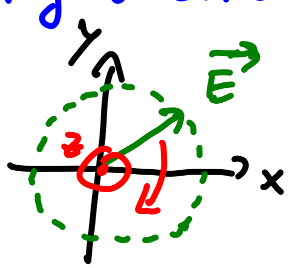

$|\vec{E}| = \text{const}$

$a = b$

$\varphi_a = \varphi_b - \underbrace{\frac{\pi}{2}}_{90^\circ}$

$|L\rangle = \frac{1}{\sqrt{2}} \{ i|x\rangle + |y\rangle \}$

5) right circular polarized:




$a = b \qquad \varphi_a = \varphi_b + \frac{\pi}{2}$

$|R\rangle = \frac{1}{\sqrt{2}} \{ (-i)|x\rangle + |y\rangle \}$

# Experiments:

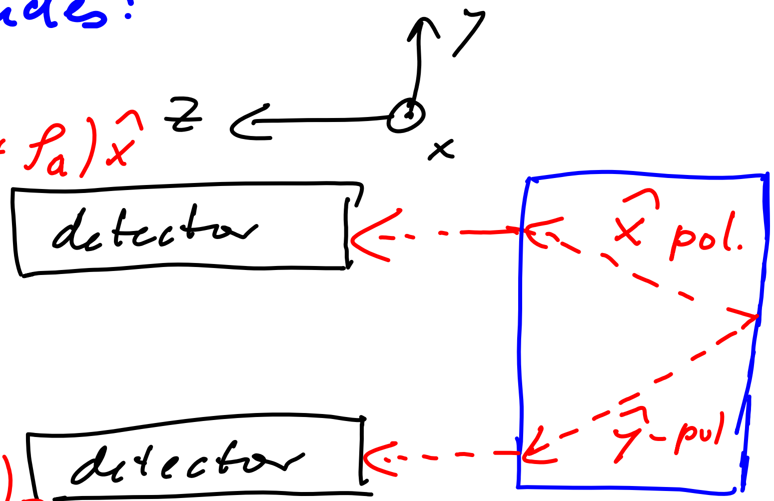
① Linear polarization analyzer and Projection amplitudes:

② Linear polarization analyzer loop/change of basis states:

③ Example of Interference of Polarization States:

④ Circular polarized light/complex projection amplitudes

## ① Linear polarization analyzer and Projection amplitudes:




$$\vec{E}_x = a\cos(kz - \omega t + \phi_a)\hat{x}$$
$$= \hat{x}(\vec{E}_i \cdot \hat{x})$$
$$I_x \propto \frac{1}{2}a^2$$

$$\vec{E}_y = b\cos(kz - \omega t + \phi_b)\hat{y}$$
$$= \hat{y}(\vec{E}_i \cdot \hat{y})$$
$$I_y \propto \frac{1}{2}b^2$$

linear polarization analyzer (calcite crystal)

incident light of some polarization

$$\vec{E}_i = a\cos(kz - \omega t + \phi_a)\hat{x} + b\cos(kz - \omega t + \phi_b)\hat{y}$$

$$I_{in} \propto \overline{|\vec{E}|^2} = \frac{1}{2}a^2 + \frac{1}{2}b^2$$

$$I_{in} = I_x + I_y$$ : no intensity lost!