Towards a proof of the Riemann Hypothesis

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Cornell Physics Dept. Colloquium
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Based on:

AL Int. J. Mod. Phys. (2008)

AL Int. J. Mod. Phys. (2013)

G. França and AL, Commun. Number Theory and Phys. 2015

G. França and AL, arXiv:1509.03643 (2015) [math.NT]

AL arXiv:1601.00914 (2016) [math.NT]

Introductory material reviewed in my Riemann Center lectures: arXiv:1407.4358

What is the RH?

Riemann zeta function was originally defined by the series:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \ldots, \qquad \Re(z) > 1$$

It can be analytically continued to the whole complex z plane.

It has an infinite number of trivial zeros: $\zeta(-2n) = 0$

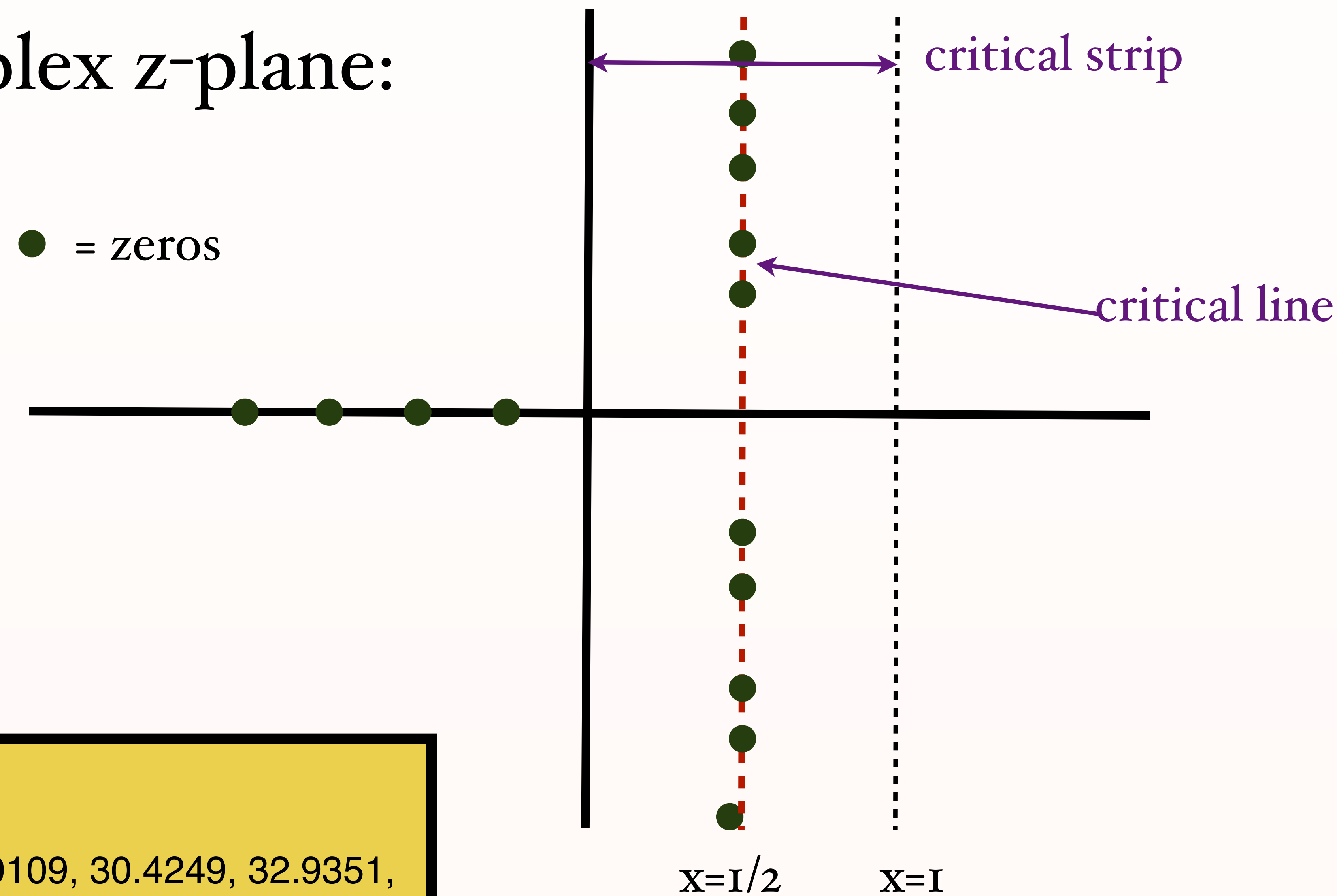
It has a simple pole at z=1.

Riemann Hypothesis: All non-trivial zeros of Zeta have real part 1/2. That is they are of the form:

$$\zeta(\rho) = 0, \qquad \rho = \frac{1}{2} + iy$$

1859

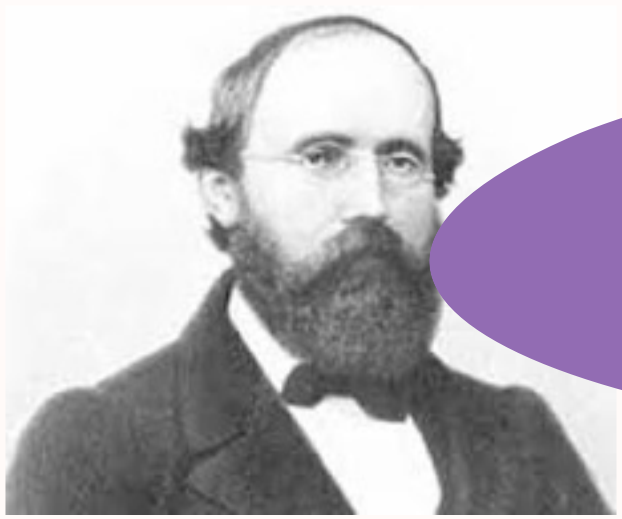
complex z-plane:



The first few:

14.1347, 21.022, 25.0109, 30.4249, 32.9351, 37.5862, 40.9187, 43.3271, 48.0052, 49.7738

“One would of course like to have a rigorous proof of this, but I have put aside the search for a proof after some fleeting vain attempts because it is not necessary for the immediate objective of my investigation.”



Remarks:

- This is a problem in analytic number theory.
- Importance of the RH: deep implications for distribution of prime numbers. Encryption etc.
- 8th of Hilbert's 23 problems (1900). "*If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?*"
- Known numerically that the first 10^{13} zeros are on the line.
- No prior well-posed strategy towards a proof.
- $ involved: one of 7 Clay Millennium prizes.

". . .there have been very few attempts at proving the Riemann hypothesis, because, simply, no one has ever had any really good idea for how to go about it."

- Selberg

Remarks on our approach:

- Main idea: RH follows from the multiplicative independence of the primes. Analogy with statistical mechanics of a large number of particles. There are many more primes than Avogadro's number: $N_A = 10^{23}$

- The approach is universal, i.e. applies to at least two infinite classes of "zeta" functions, Dirichlet L-functions and those based on cusp (modular) forms such as Ramanujan tau L-function.

- There are no logical gaps, but at least a few results need more rigor to be up to standards of modern pure math. Will indicate where with [*delicate].

- It's a "constructive" approach, i.e. leads to new formulas, etc.

- For instance, I calculated the 10^{100} -th zero:

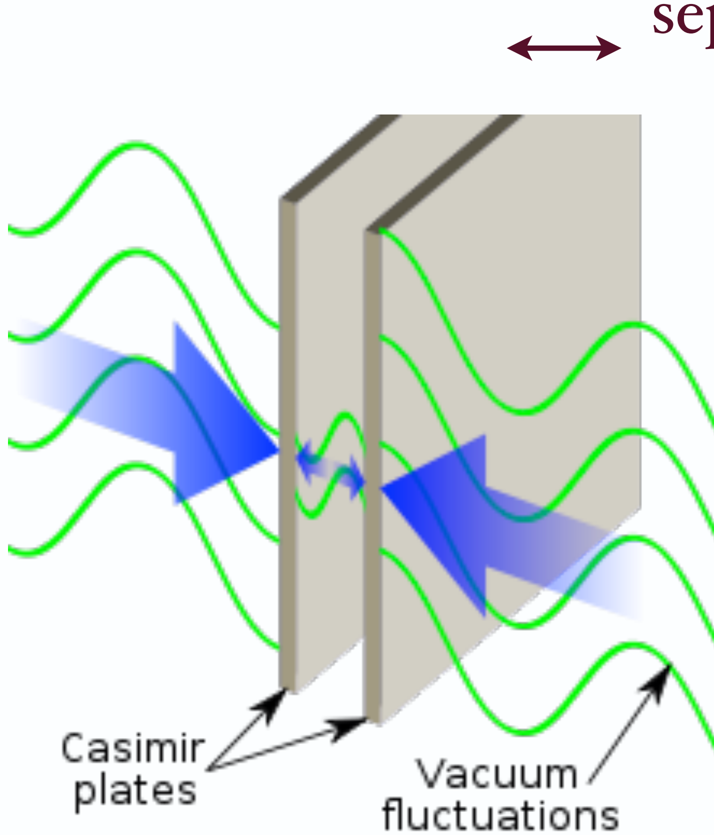
$$
\begin{aligned}
t_n = {} & 28069038384289406990319544583825640008454803016284 6 \\
& 0451923600592249309223490730430603356531092524 73.244\ldots
\end{aligned}
$$

Outline

- A bit of physics, zeta in Casimir effect, blackbody radiation, BEC.
- Generalized RH: Dirichlet series.
- Our main theorem: Random Walks and central limit theorems.
- Transcendental equations for individual zeros and a second theorem.
- Computing very high zeros from the primes.

The Casimir effect and Zeta

separation = ℓ



Force= $F(\ell) = -dE_{\rm vac}(\ell)/d\ell.$

energy density: $$\rho_{\rm vac}^{\rm cas} = -\pi^2/720\ell^4.$$

This effect has been measured.

For now note: 720 = 6 x 120

Cylindrical version of Casimir effect

Just change boundary conditions: join plates at edges to have periodic b.c.

One compactified spatial dimension of circumference β

2 dim'l space + time

Relation to Casimir: $\rho_{\rm vac}^{\rm cas}(\ell) = 2\rho_{\rm vac}^{\rm cyl}(\beta = 2\ell)$

$$\rho_{\rm vac}^{\rm cyl} = \frac{1}{2\beta}\sum_{n\in\mathbb{Z}}\int\frac{d^2\mathbf{k}}{(2\pi)^2}\sqrt{\mathbf{k}^2+(2\pi n/\beta)^2} = -\beta^{-4}\pi^{3/2}\Gamma(-3/2)\zeta(-3) + \text{const.}$$

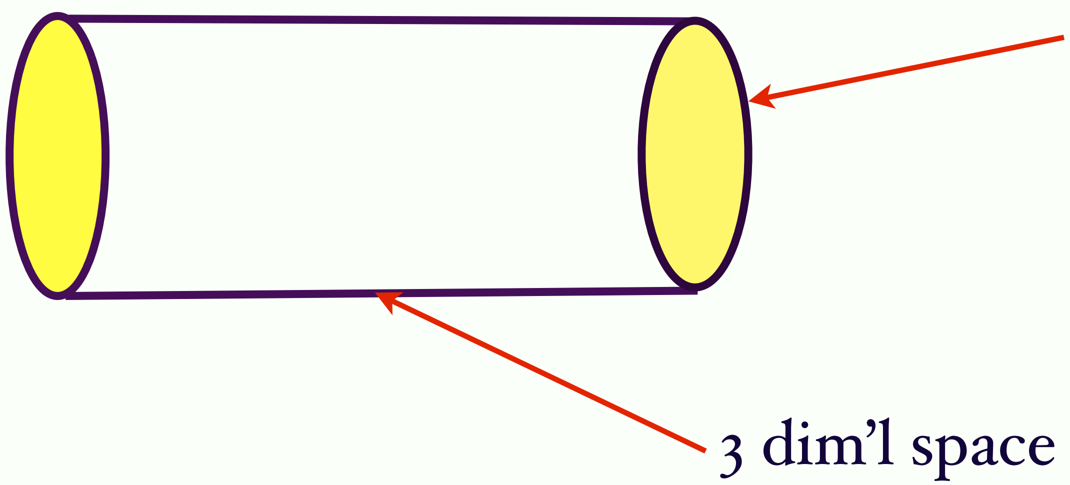
quantized modes on circle

$$\zeta(-3) = 1 + 2^3 + 3^3 + 4^3 \ldots = \quad ??$$

This is related to the Cosmological constant problem.

Quantum Statistical Mechanics viewpoint.

Passing to euclidean time $t = -i\,\tau$, ϱ_{vac} is just the finite temperature free energy on the cylinder with circumference $\beta = 1/T$.



Quantum Statistical. Mech.
gives a very different
convergent expression.

$$\rho_{\rm vac}^{\rm cyl} = \frac{1}{\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\log\left(1 - e^{-\beta k}\right) = -\beta^{-4}\frac{\zeta(4)}{2\pi^{3/2}\Gamma(3/2)} = -\frac{\pi^2}{90}T^4.$$

black body

These two calculations must give the same result:

$$\frac{\zeta(4)}{2\pi^{3/2}\Gamma(3/2)} = \pi^{3/2}\Gamma(-3/2)\zeta(-3) \qquad ??$$

YES!
Due to the amazing functional equation:

$$\chi(z) \equiv \pi^{-z/2}\Gamma(z/2)\zeta(z) = \chi(1-z)$$

(proven by Riemann)

Nature knows about analytic continuation:

$$\zeta(-3) = 1 + 2^3 + 3^3 + 4^3 + \ldots\ldots = \ ?$$
$$= \frac{1}{120}$$

By analytic continuation!

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Zeta and Bose Einstein Condensation

The critical point for BEC in d spatial dimensions satisfies:

$$n_c \lambda_T^d = \zeta(d/2) \qquad \lambda_T = \text{thermal de Broglie wavelength} \; = \hbar\sqrt{2\pi/mk_B T}$$

BEC is not possible in d=2 dimensions:

$$\text{pole at } z = 1: \quad \zeta(1) = \infty$$

In physics, this is a manifestation of the Honenberg-Coleman-Mermin-Wagner theorem. In number theory this is a consequence of there being an infinite number of primes.

The distribution of Prime Numbers and Zeta

Prime number theorem

How many primes less than x?

Gauss, a 15 years old boy, guessed in 1792

$$\pi(x) = \sum_{p \leq x} 1 \approx \frac{x}{\log x} \approx \mathrm{Li}(x)$$

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}$$

- Chebyshev (1850) tried to prove using $\zeta(z)$
- Only proven 100 years later (1896) by Hadamard/de la Vallé Poussin $\boxed{\zeta(1+iy) \neq 0}$

Zeta and the Primes ....

The Golden Key: Euler product formula:

(1737)

$$\zeta(z) = \prod_{n=1}^{\infty} \frac{1}{1 - p_n^{-z}}$$

$p_n = \mathrm{n-th\ prime}$

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \dots$$

Sieve method:

$$\frac{1}{2^z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \frac{1}{6^z} + \dots$$

$$(1 - \frac{1}{2^z})\zeta(z) = 1 + \frac{1}{3^z} + \frac{1}{5^z} + \dots$$

$$(1 - \frac{1}{3^z})(1 - \frac{1}{2^z})\zeta(z) = 1 + \frac{1}{5^z} + \frac{1}{7^z} + \dots$$

Remarks:

1. Pole at z=1 implies there are an infinite number of primes (recall BEC).

2. There are no zeros with Re(z)> 1 due to EPF. (will be important).

Riemann's Main Result

$$\pi(x) = \sum_{n \geq 1} \frac{\mu(n)}{n} J(x^{1/n}).$$

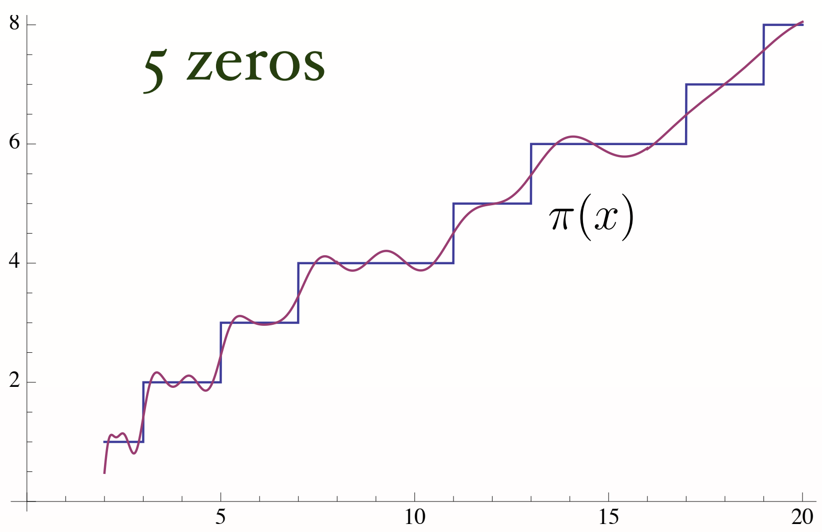
$$J(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}\left(x^{\rho}\right) + \int_{x}^{\infty} \frac{dt}{\log t} \, \frac{1}{t\left(t^{2}-1\right)} - \log 2,$$

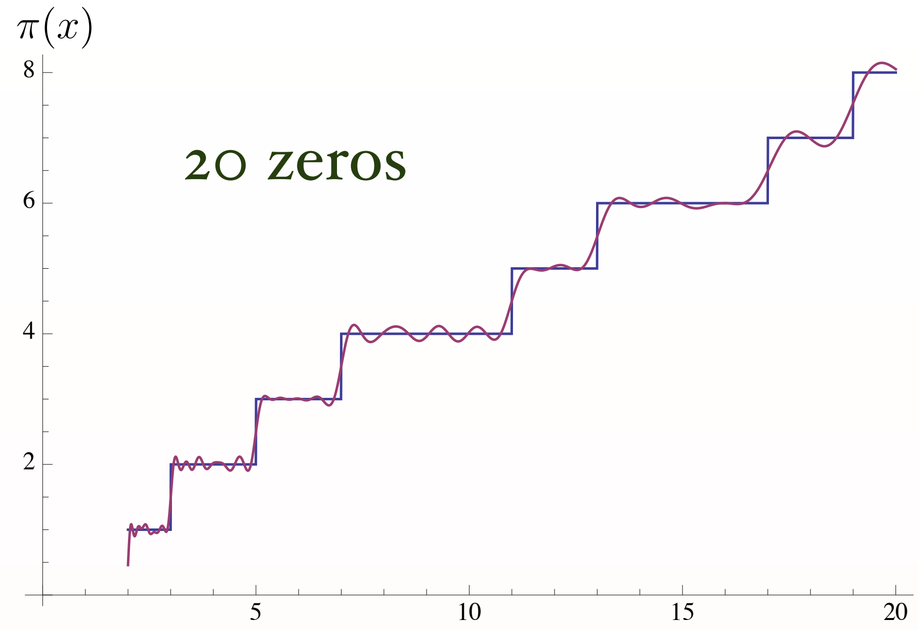
ϱ = a zero on the critical strip

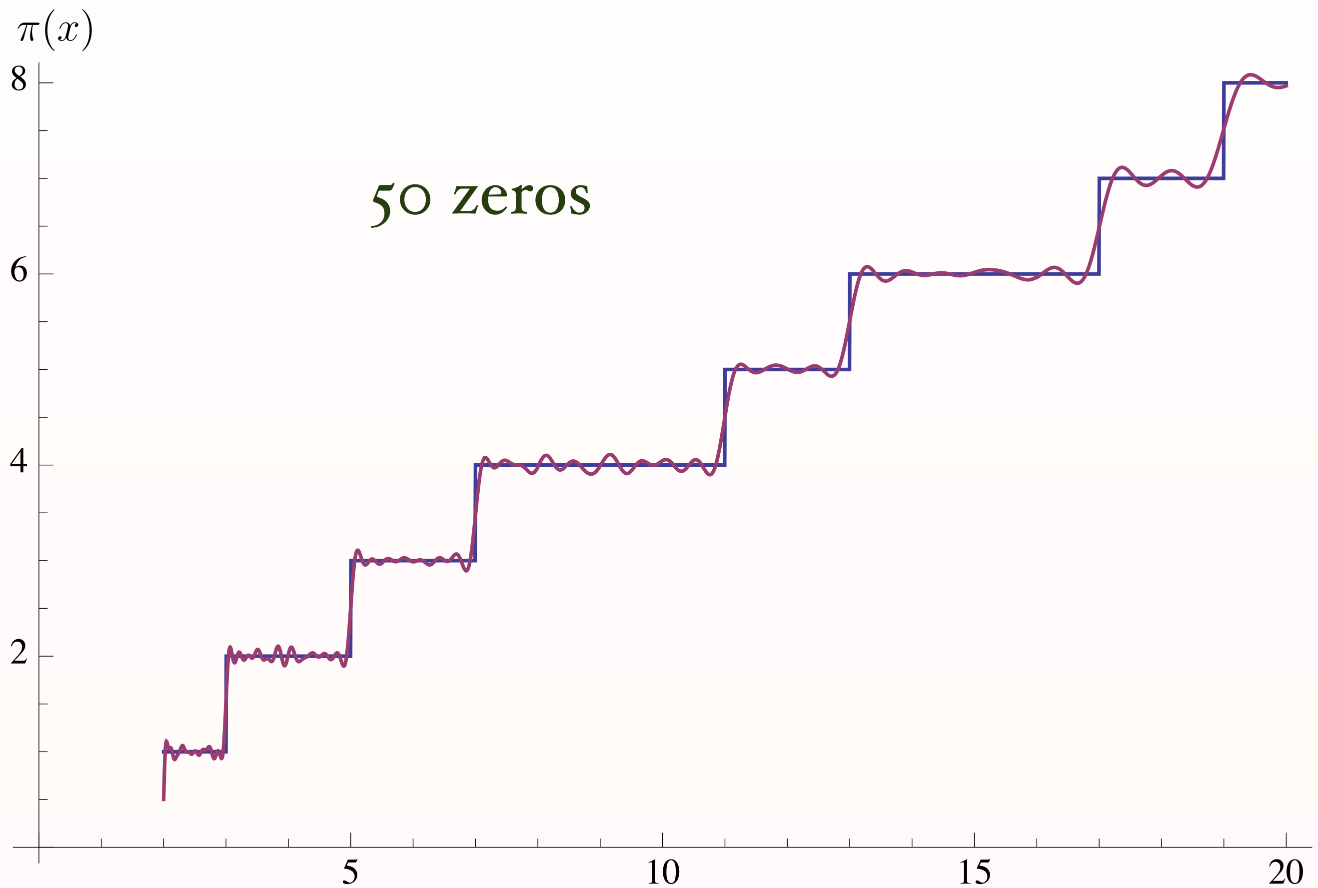
Derived using clever real and complex analysis.

Here, $\mu(n)$ is the Möbius function, equal to 1 (-1) if n is a product of an even (odd) number of distinct primes, and equal to zero if it has a multiple prime factor. The above expression is actually a finite sum, since for large enough n, $x^{1/n} < 2$ and $J = 0$.

Remark: if there are no zeros with real part equal to 1, Li(x) is the leading term. That's how the prime number theorem was proven.







To calculate primes, you need to calculate zeros. We obtained the converse, as we will see.

Mysteries of the Primes

How can the ordered set of the integers

{1,2,3,4,5, ….}

give rise to the seemingly random series of primes:

{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, …}?

"God may not play dice with the universe, but something strange is going on with the prime numbers." - Pomerance/Erdos

"...there is no apparent reason why one number is prime and another not. To the contrary, upon looking at these numbers one has the feeling of being in the presence of one of the inexplicable secrets of creation." Zagier 1977

"It is evident that the primes are randomly distributed but, unfortunately, we don't know what 'random' means." Vaughn 1990

We are going to use this pseudo-randomness of the primes to our advantage.

Generalized RH: Dirichlet L-functions

Arithmetic Dirichlet characters of modulus k:

Axiomatic definition:

1. $\chi(n+k) = \chi(n)$. (periodicity)
2. $\chi(1) = 1$ and $\chi(0) = 0$.
3. $\chi(nm) = \chi(n)\chi(m)$. (multiplicativity, the most important)
4. $\chi(n) = 0$ if $(n,k) > 1$ and $\chi(n) \neq 0$ if $(n,k) = 1$.
5. If $(n,k) = 1$ then $\chi(n)^{\varphi(k)} = 1$, where $\varphi(k)$ is the Euler totient arithmetic function. This implies that $\chi(n)$ are roots of unity. (roots of unity)

k=3 example:

n	1	2	3	4	5	6	7
$\chi(n)$	1	-1	0	1	-1	0	etc

k=7 Example:

n	1	2	3	4	5	6	7
$\chi_{7,2}(n)$	1	$e^{2\pi i/3}$	$e^{\pi i/3}$	$e^{-2\pi i/3}$	$e^{-\pi i/3}$	-1	0

L-function: $$L(z,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^z} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^z}\right)^{-1}$$

$p_n = \mathrm{n-th\ prime}$

Also satisfies functional eqn. relating L(z) to L(1-z)

Generalized RH

Our Main Theorem:

change to standard notation : $z \to s = \sigma + it$

Consider the series: $$B_N = \sum_{n=1}^{N} \cos(\lambda_n)$$

$$\lambda_n = t \log p_n - \arg \chi(p_n)$$

Theorem: ***If*** $B_N = O(\sqrt{N})$ ***then the RH is true, since the EPF is valid for Re(s)>1/2***

* The significance of Re(s) > 1/2, i.e. right half of critical strip, arises from this square root,

* Why would $B_N = O(\sqrt{N})$? Because it behaves like a RANDOM WALK due to the multiplicative independence of the primes.

The Random Walk Property

Let t=0 for non-principal Dirichlet character:

$$B_N = \sum_{n=1}^{N} \cos(\theta_{p_n}), \qquad \theta_{p_n} = \arg \chi(p_n)$$

* The angles θ_{p_n} are discrete and equally spaced on unit circle.

* The series behaves like a sum of independent, identically distributed random variables, i.e. a random walk [*delicate].

* Example of the k=3 character:

$$\chi(n) \text{ over integers n} : \{1, -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, 0, 1, -1, \ldots\}$$

$$\chi(p_n) \text{ over primes } \mathrm{p_n} : \{-1, 0, -1, 1, -1, 1, -1, 1, -1, -1, 1, 1, -1, 1, -1, -1, -1, 1, 1, \ldots\}$$

The latter looks random! Some properties can be proven.

The original Riemann Zeta case

Actually more subtle than non-principal Dirichlet. All angles are zero and one has to consider:

$$B_N = \sum_{n=1}^{N} \cos(t \log p_n)$$

Theorem of Kac (1959) nearly does the job (proven at Cornell):

$B_N/\sqrt{N}$ has a normal distribution in the limit $t \to \infty$

One needs a bit more (finite t), which I won't discuss here since it's a bit technical. [*delicate]

To be cautious, (although I believe we have a proof), we will only Conjecture that:

$$B_N = O(\sqrt{N})$$

(the crux of our theory, the only thing left to rigorously prove. [*delicate]

"there must be something mysterious about the normal law since mathematicians think it is a law of nature whereas physicists are convinced that it is a mathematical theorem." -POINCARE

Sketch of proof of main theorem:

One just needs to prove the EPF is valid to right of the critical line:

Theorem 2. *For $\sigma > \frac{1}{2}$,*

$$\lim_{t\to\infty} \zeta(\sigma+it) = \lim_{N\to\infty}\lim_{t\to\infty}\prod_{n=1}^{N}\left(1-\frac{1}{p_n^{\sigma+it}}\right)^{-1} \tag{14}$$

Proof. The proof is essentially the same as in [1, 2], so we just sketch the main steps involved. Taking the logarithm of the above equation, one concludes that the Euler product converges with $\sigma > \frac{1}{2}$ if the series $X_N(s) = \sum_{n=1}^{N} 1/p_n^s$ converges as $N \to \infty$. It is enough to consider $\mathcal{S}_N = \Re(X_N)$:

$$\mathcal{S}_N(s) = \sum_{n=1}^{N} a_n b_n, \qquad a_n = \frac{1}{p_n^\sigma}, \qquad b_n = \cos(t\log p_n) \tag{15}$$

The latter can be reorganized using an Abel transform, i.e. summation by parts:

$$\mathcal{S}_N = a_N B_N + \sum_{n=1}^{N-1} B_n(a_n - a_{n+1}), \quad B_n \equiv \sum_{k=1}^{n}\cos(t\log p_n) \tag{16}$$

The sum above is bounded

$$|\mathcal{S}_N| \le \sigma\sum_{n=1}^{N-1}|B_n|\frac{g_n}{p_n^{\sigma+1}} + O(1) \tag{17}$$

where $g_n = p_{n+1} - p_n$ is the gap between primes. One then performs another summation by parts using a summed version of the Cramér-Granville conjecture

$$\sum_{n=1}^{N} g_n < \sum_{n=1}^{N}\log^2 p_n \tag{18}$$

The latter was proven in [6],[2]. Now if $\lim_{t\to\infty} B_N(t) = O(\sqrt{N})$ for large N, as far as convergence is concerned, the sum in (17) behaves as $\sum_n \log^2 n/n^{\sigma+1/2}$ which converges for $\sigma > \frac{1}{2}$. □

Used prime # thm. $p_n \approx n\log n$

Numerical Evidence is compelling.

Random Walk property:

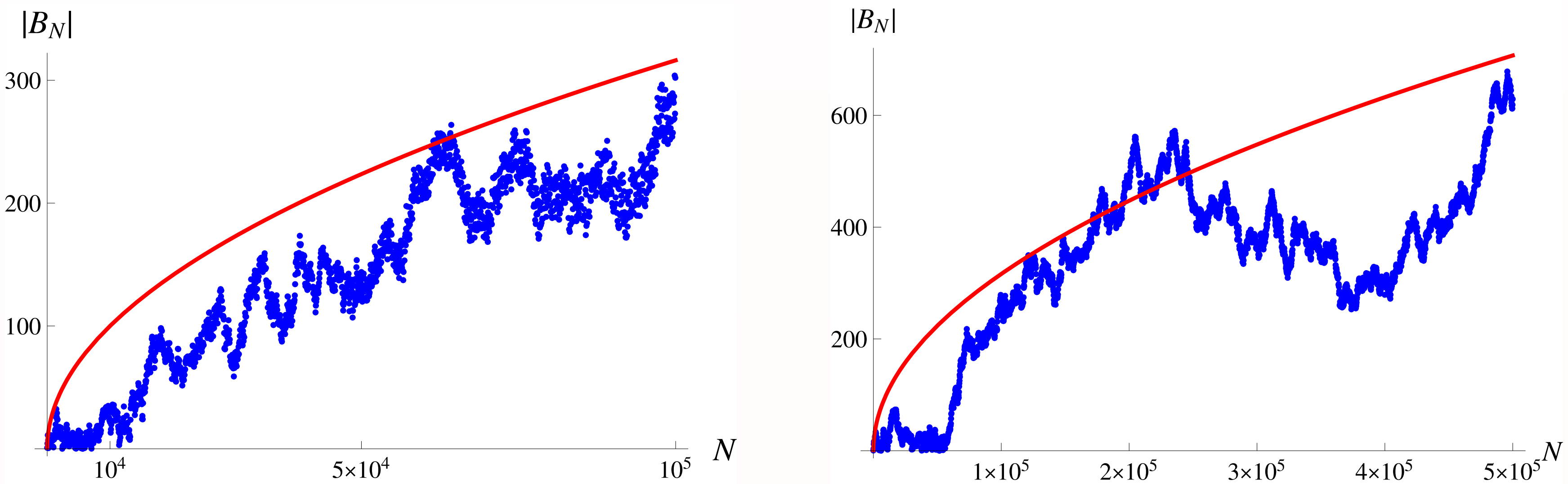


FIG. 1. The absolute value of the partial sum B_N versus N, for a fixed t. **Left:** We use (23) with $t = 5 \cdot 10^3$. Note that N is below the cut-off (30). **Right:** Here we use (21) ($u = 1$) with the character $\chi = \chi_{7,2}$ shown in (A3), and $t = 5 \cdot 10^2$. In this case we can freely take the limit $N \to \infty$.

Convergence of EPF, next slide:

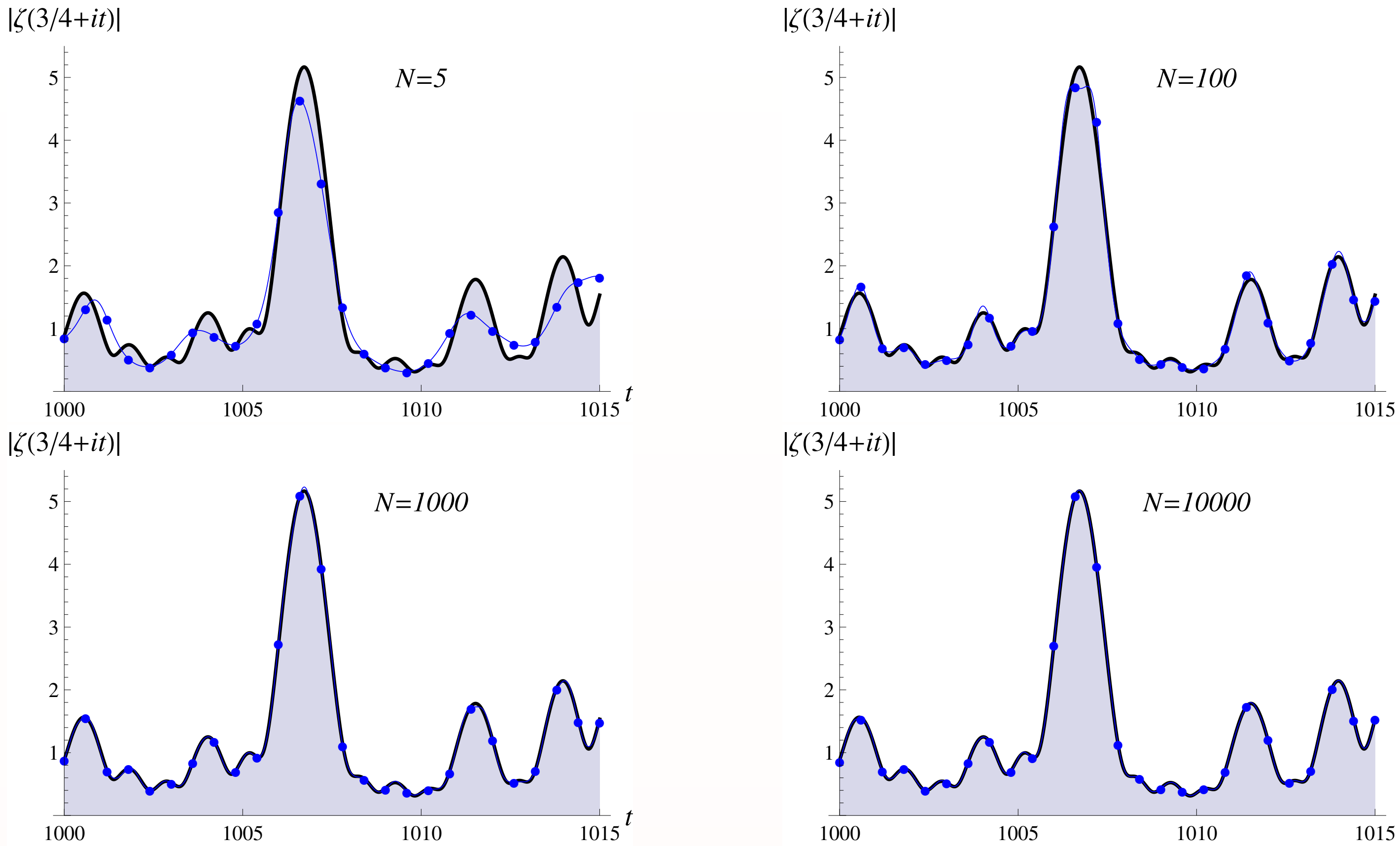


FIG. 4. The black line is the actual $|\zeta(3/4+it)|$, analytically continued into the strip, and the blue line is the partial product $|\mathcal{P}_N(3/4+it)|$. Dots are added to the line to aid visualization.

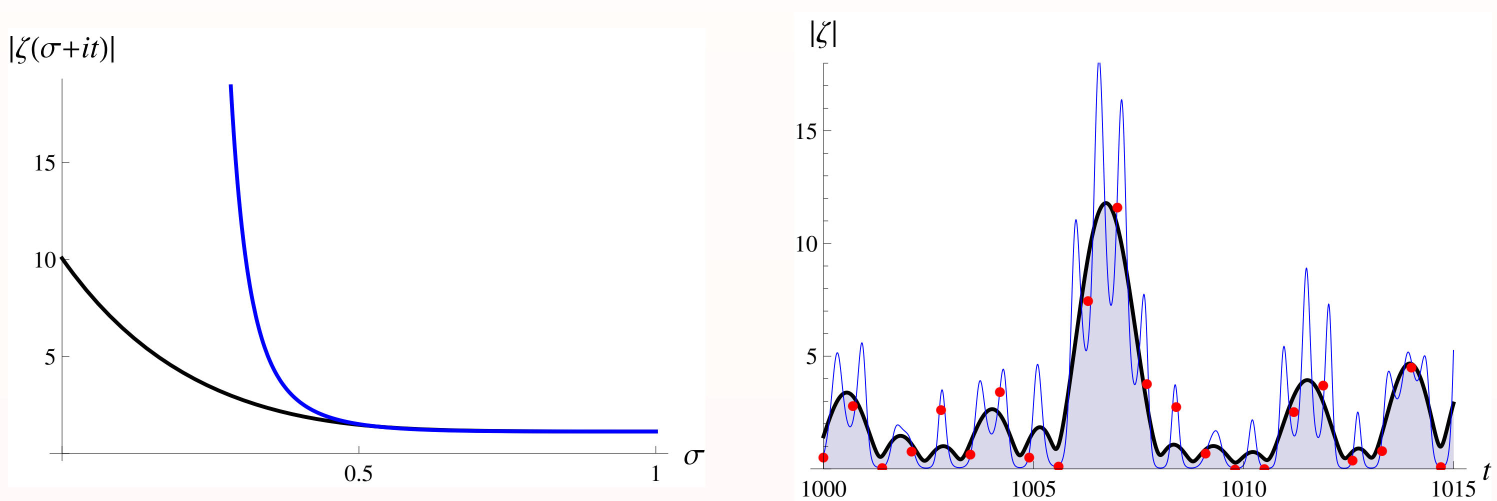


FIG. 6. **Left:** the black line corresponds to $|\zeta(\sigma+it)|$ against $0<\sigma<1$, for $t=500$. The blue line is the partial product $|\mathcal{P}_N(\sigma+it)|$ with $N=10^4$. **Right:** the black line is the exact $|\zeta|$, and the blue line is the partial product $|\mathcal{P}_N|$ (with $N=8\cdot 10^3$), against t. We took $\sigma=0.4$. The red dots are the Cesàro average $|\langle\mathcal{P}_N\rangle|$. If we increase N the results are even worse.

Transcendental equations for individual zeros.

AL Int. J. Mod. Phys. A28 (2013)
G. França, AL, Comm. Numb. Theory and Phys. 2015

Everyone here knows one function with an infinite number of zeros along a line in the complex z-plane........

$\cos(z) = 0$
for $z = (n+1/2)\pi$

The single equation $\zeta(\rho) = 0$ has an infinite number of solutions. We replace it with an infinite number of equations, one for each zero, in one-to-one correspondence with zeros of a cosine.

The n-th zero satisfies a Transcendental Equation that depends only on n.

How to quickly derive this equation (omitting some details):

$$\text{Recall}: \quad \chi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \chi(1-s)$$

$$\text{If } \chi(\rho) = 0, \text{ then } \quad \chi(\rho) + \chi(1-\rho) = 0$$

$$\text{write} \quad \chi = Ae^{i\phi}$$

$$\text{on critical line}: \qquad \cos\phi = 0 \Longrightarrow \phi = (n - \tfrac{3}{2})\pi$$

$$\text{n} - \text{th zero}: \qquad \rho_n = \tfrac{1}{2} + it_n$$

$$\phi = \Im \log \Gamma\left(\tfrac{1}{4} + it_n/2\right) - t_n \log\sqrt{\pi} + \lim_{\delta\to 0^+} \arg\zeta(\tfrac{1}{2} + \delta + it_n) = (n - \tfrac{3}{2})\pi$$

Important: For arg = the phase, one must keep track of the branches, i.e. it is on a multi-sheeted Riemann surface.

Solving the exact version of the transcendental equation gives zeros to any desired accuracy.

The 1000-th zero to 500 digits:

1419.422480945995686465989038079916819232100601064166016304690814684608676417593010417911343291179209987480984232260560118741397447952650637067250834288983151845447688252593115944239425195484687708163946256332381457791528418559343151187932905776427998012736052409446117337041818962494747459675690479839876840142804973590017354741319116293486589463954542313208105699019807193917543029984881490193193671823126420427276358911487848329996467356160858436515425171824179566414953524432921936494838577722534600 88

..........with very simple Mathematica commands.

Another Theorem: If there is a unique solution to this equation for every n, then the RH is true and all zeros are simple.

PROOF: If there is a unique solution to this equation for every n, since they are enumerated by n, we can count how many zeros are on the critical line up to a height t=T.

$N_0(T)$ = number of zeros on the line with ordinate t<T. The above formula implies (for large T):

$$N_0(T) = \frac{T}{2\pi}\log\left(\frac{T}{2\pi e}\right) + \frac{7}{8} + \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2}+iT\right) + O\left(T^{-1}\right)$$

Now: N(T) = number of zeros on *the entire critical strip* has been known for over 100 years by performing a certain contour integral (*argument principle)* around the strip (Riemann, Backlund).

our $N_0(T)$ = the known N(T)

Thus: all zeros are on the line **if one can prove there is a unique solution**. The EPF can be used to show this. [*delicate]

Calculating very high zeros from primes

Recall Riemann's main result: to calculate primes, one needs to know the zeros of zeta.

We can obtain the converse: to calculate zeros, you need to know all the primes!

HOW: Use the Stirling approximation for log Gamma,
and the Euler Product Formula for arg Zeta.

Let $t_{n:N}$ denote the n-th zero computed using N primes. (ordinate)

$$\frac{t_{n;N}}{2}\log\left(\frac{t_{n;N}}{2\pi e}\right)-\frac{\pi}{8}+\frac{1}{48t_{n;N}}-\lim_{\delta\to0^+}\Im\sum_{k=1}^{N}\log\left(1-\frac{1}{p_k^{1/2+\delta+it_{n;N}}}\right)=(n-\tfrac{3}{2})\pi$$

Every individual zero knows about all the primes!

If N=0 primes, there is a unique solution in terms of the Lambert W function:

$$t_{n;0} = \widetilde{t}_n \equiv \frac{2\pi(n - \frac{11}{8})}{W[(n - \frac{11}{8})/e]}$$

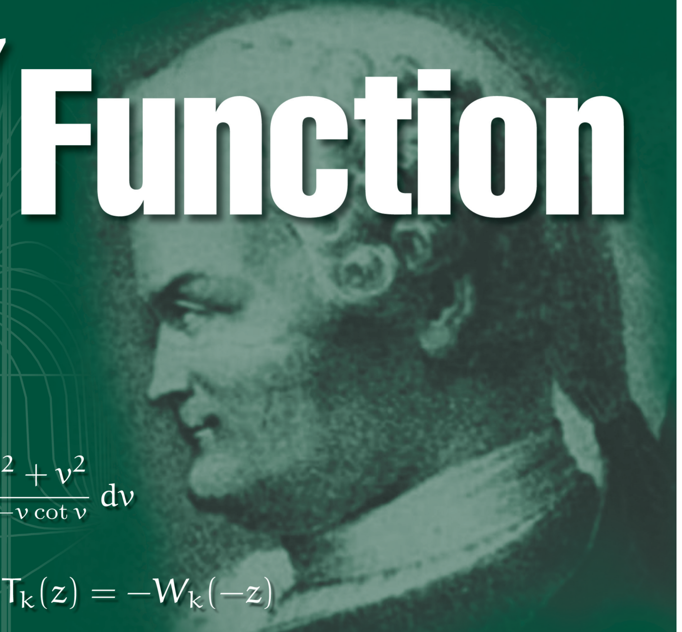
(previously unknown)

W is defined to satisfy: $W(z)e^{W(z)} = z$

Lambert W was first studied by Lambert in the 1758. Euler recognized its importance in 1779 in a paper on transcendental equations, and credited Lambert. He was the first to prove Pi is irrational (Euler tried), and introduced hyperbolic functions like cosh.

It's importance was only realized in the 1990's, when it finally obtained the name
the Lambert W-function.

The Lambert W Function

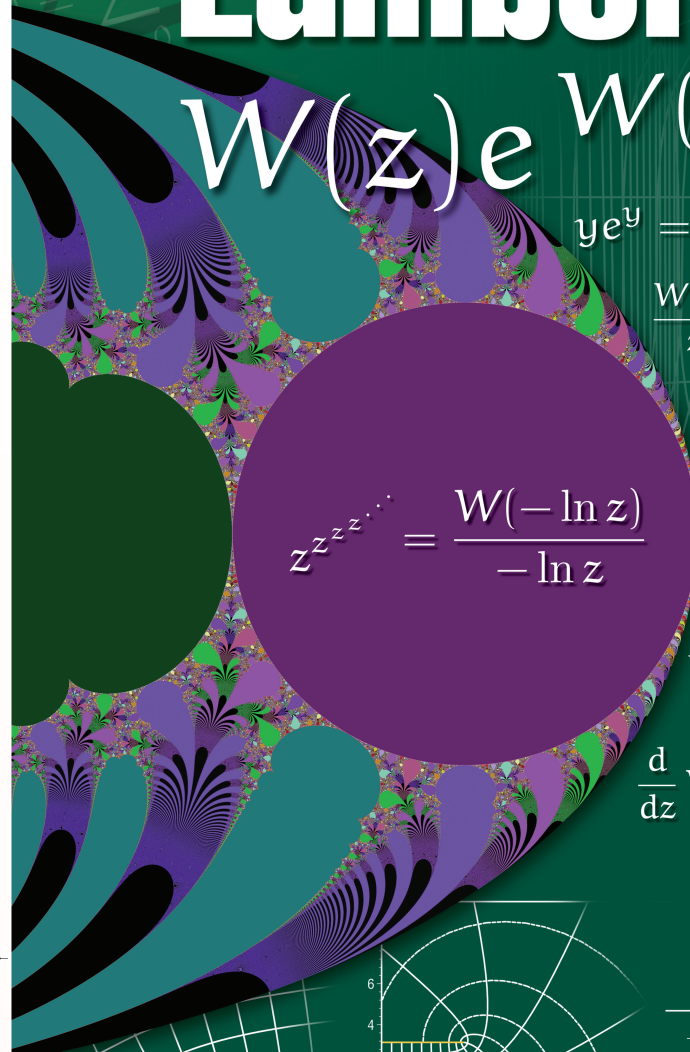


$$W(z)e^{W(z)} = z$$

$$ye^{y} = z \iff y = W_k(z)$$

$$\frac{W(z)}{z} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1 - v\cot v)^2 + v^2}{z + v\csc v\, e^{-v\cot v}}\, dv$$

$$ye^{-y} = z \iff y = T_k(z) = -W_k(-z)$$



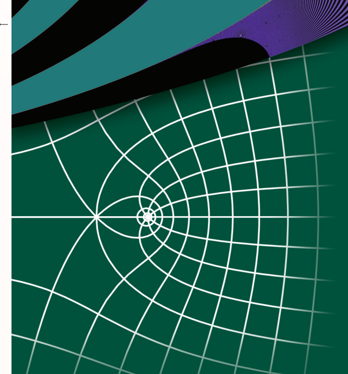
$$z^{z^{z^{z^{\cdot^{\cdot^{\cdot}}}}}} = \frac{W(-\ln z)}{-\ln z}$$

A Fractal Related to W

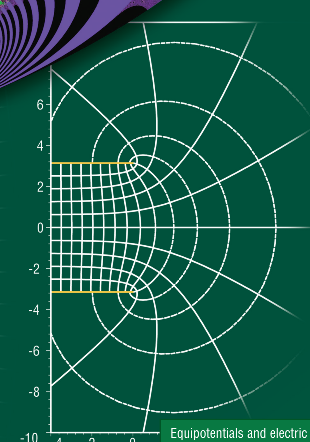
Each colour represents a cycle length in the iteration $a_{n+1} = z^{a_n}$, with $a_0 = 1$. A pixel at coordinate $\zeta = x + iy$ where $\zeta = T(\ln z)$ is given the colour corresponding to the length of the attracting cycle.

$$W(z) = \sum_{n \geq 1} \frac{(-n)^{n-1}}{n!} z^n$$

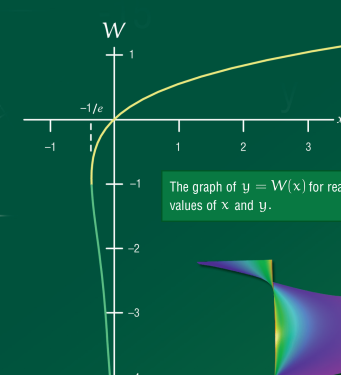
$$\frac{d}{dz} W(z) = \frac{W(z)}{z(1 + W(z))} \quad \text{if } z \neq 0, -1/e$$



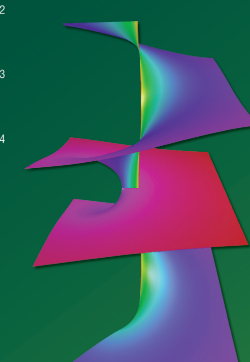
Images of circles and rays under the maps $z \mapsto W_k(z)$. Equivalently, images of horizontal and vertical lines under the map $z \mapsto \omega(z) = W_{K(z)}(e^z)$.



Equipotentials and electric field lines at the edge of a capacitor consisting of two charged thin plates a distance 2π apart.
$\zeta = z - 1 + \omega(z - 1)$



The graph of $y = W(x)$ for real values of x and y.



A portion of the Riemann surface for $W(z)$, drawn by plotting a surface with height $\mathrm{Im}\,(W(x+iy))]$ at coordinates (x, y) and colouring the surface with Re $(W(x+iy))$; the apparent intersection on the line $-1/e \leq x \leq 0,\ y = 0$ is of surfaces with different colours and therefore not a true intersection.

$$\int W(z)\, dz = \frac{z\left(W^2(z) - W(z) + 1\right)}{W(z)} + C$$

$$\int_0^\infty x^{s-1} W(x)\, dx = \frac{(-s)^{-s}\,\Gamma(s)}{s} \quad \text{if } -1 < \mathrm{Re}(s) < 0$$

$$\int 2\sin W(x)\, dx = \left(x + \frac{x}{W(x)}\right)\sin W(x) - x\cos W(x) + C$$

$$\int_0^\infty e^{-st} W(e^t)\, dt = s^{s-2}\Gamma(1 - s, sW(1)) + \frac{W(1)}{s} \quad \text{if } \mathrm{Re}(s) > 0$$

Johann Heinrich Lambert

Johann Heinrich Lambert was born in Mulhouse on the 26th of August, 1728, and died in Berlin on the 25th of September, 1777. His scientific interests were remarkably broad. The self-educated son of a tailor, he produced fundamentally important work in number theory, geometry, statistics, astronomy, meteorology, hygrometry, pyrometry, optics, cosmology and philosophy. Lambert was the first to prove the irrationality of π. He worked on the parallel postulate, and also introduced the modern notation for the hyperbolic functions.

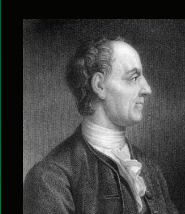
In a paper entitled “Observationes Variae in Mathesin Puram”, published in 1758 in *Acta Helvetica*, he gave a series solution of the trinomial equation, $x^m + px = q$, for x. His method was a precursor of the more general Lagrange inversion theorem. This solution intrigued his contemporary, Euler, and led to the discovery of the Lambert W function.

Lambert wrote Euler a cordial letter on the 18th of October, 1771, expressing his hope that Euler would regain his sight after an operation; he explains in this letter how his trinomial method extends to series reversion.

The Lambert W function is *implicitly elementary*. That is, it is implicitly defined by an equation containing only elementary functions. The Lambert W function is not, itself, an elementary function. It is also not a *Liouvillian* function, which means that it is not expressible as a finite sequence of exponentiations, root extractions, or antidifferentiations (quadratures) of any elementary function.

The Lambert W function has been applied to solve problems in the analysis of algorithms, the spread of disease, quantum physics, ideal diodes and transistors, black holes, the kinetics of pigment regeneration in the human eye, dynamical systems containing delays, and in many other areas.

Leonhard Euler



Leonhard Euler was born on the 15th of April, 1707, in Basel, Switzerland, and died on the 18th of September, 1783, in St. Petersburg, Russia. Half his papers were written in the last fourteen years of his life, even though he had gone blind.

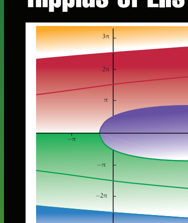
Euler was the greatest mathematician of the 18th century, and one of the greatest of all time. His work on the calculus of variations has been called “the most beautiful book ever written”, and Pierre Simon de Laplace exhorted his students: “Lisez Euler, c’est notre maître â tous”, advice that is still profitable today.

Many functions and concepts are named after him, including the Euler totient function, Eulerian numbers, the Euler-Lagrange equations, and the “eulerian” formulation of fluid mechanics. The mathematical formulae on this poster are typeset in the Euler font, designed by Hermann Zapf to evoke the flavour of excellent human handwriting.

Lambert’s series solution of his trinomial equation, which Euler rewrote as $x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}$, led to the series solution of the transcendental equation $x \ln x = v$. This was the earliest known occurrence of the series for the function now called the Lambert W function.

$$x^y = y^x \iff y = -\frac{x}{\ln x} W_k\left(-\frac{\ln x}{x}\right)$$

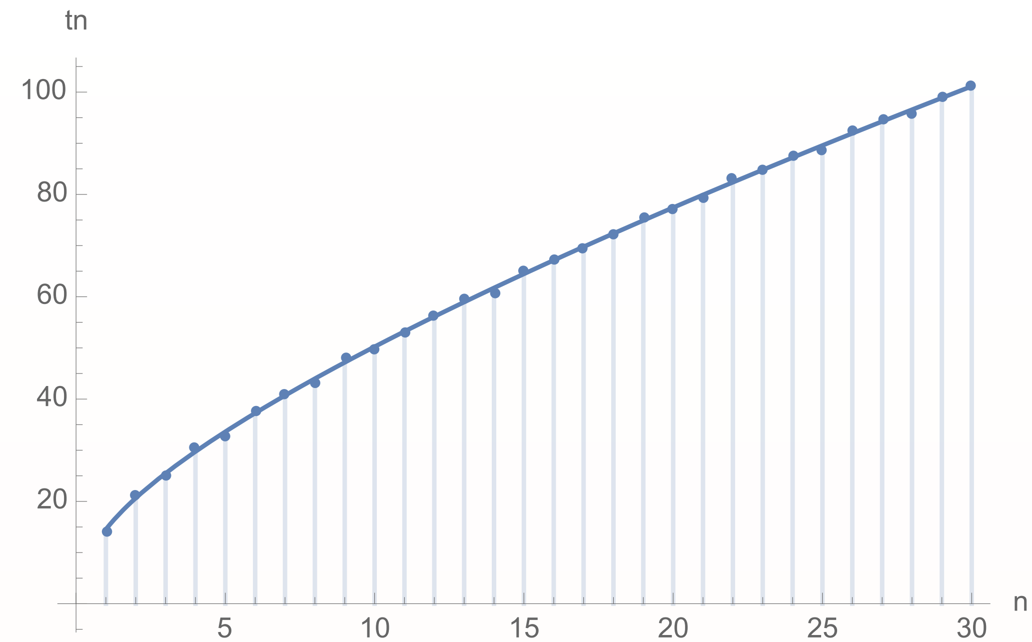
Hippias of Elis



Hippias of Elis lived, travelled and worked around 460 BC, and is mentioned by Plato. The Quadratrix (or trisectrix) of Hippias is the first curve ever named after its inventor. As drawn in the picture here, its equation is $x = -y\cot y$. This curve can be used to square the circle and to trisect the angle. Since these classical problems are unsolvable by straightedge and compass, we therefore conclude that the construction of the Quadratrix is impossible under that restriction. The Quadratrix is also the image of the real axis under the map $z \mapsto W_k(z)$, and the parts of the curve corresponding to the negative real axis delimit the ranges of the branches of W. We have here coloured the ranges of the different branches of W with different colours.

Sir Edward Maitland Wright $\omega(z) = W_{K(z)}(e^z)$

Lambert approximation:



Check high zeros with a million primes:

n	$t_{n;N}$	t_n (Odlyzko)
$10^{21} - 1$	144176897509546973538.205	$\sim .225$
10^{21}	144176897509546973538.301	$\sim .291$
$10^{21} + 1$	144176897509546973538.505	$\sim .498$
$10^{22} - 1$	1370919909931995308226.498	$\sim .490$
10^{22}	1370919909931995308226.614	$\sim .627$
$10^{22} + 1$	1370919909931995308226.692	$\sim .680$

The googol-th zero:

$n = 10^{100}\text{−th zero}:$

$$t_n = 2806903838428940699031954458382564000845480301628460451923600592249309223490730430603356531092524 73.244\ldots.$$

Conclusions

- The validity of the RH appears to need both the EPF and the functional equation.
- These two work together: The validity of the EPF and existence of solutions to the transcendental equations are closely related.
- Known counter-examples to RH have no EPF, and there are no solutions of the transcendental equation for all n.
- We extended to another infinite class of L-functions based on modular forms. Brings in reasonably recent (1975) results of Deligne in his proof of the Weil conjectures.
- A unified perspective on different of L-functions
- Only thing left to rigorously prove is the random walk property $B_N = O(\sqrt{N})$

$$\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{\left(\frac{z}{e^{z}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}\right)}}$$