1) Derive Child-Langmuir formula.

Solution:
We assume one-dimensional problem. Potential satisfies Poisson equation

$$\frac{d^2V}{dz^2} = -\frac{\rho}{\varepsilon_0}.$$

Current density and charge density are related by $J_z = \rho v_z$, while the velocity is found through energy conservation

$$\frac{1}{2}mv_z^2 = eV .$$

Eliminating ρ and v_z, Poisson distribution is rewritten as

$$\frac{d^2V}{dz^2} = -\frac{J_z}{\varepsilon_0}\sqrt{\frac{m}{2eV}}.$$

First, we need to determine if J_z depends on the coordinate. From charge conservation we have

$$\frac{\partial\rho}{\partial t} + \frac{\partial J_z}{\partial z} = 0,$$

thus, $J_z = const$ in steady state ($\partial\rho/\partial t = 0$). Therefore, one solves the differential equation for V. Use sample solution $V(z) = Az^B$ (note that $V(0) = 0$, and $dV(0)/dz = -E_z = 0$, or field at the cathode vanishes, just like Child law argues), substituting and solving for constants (Note: the value of B is obvious from the fact that second derivative proportional to $z^{-1/2}$, and $(z^B)'' = z^{B-2}$) yields

$$V = \left(-\frac{9J_z}{4\varepsilon_0}\sqrt{\frac{m}{2e}}\right)^{2/3} z^{4/3}, \text{ or } J_z = -\frac{4\varepsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{z^2}.$$

2) Prove the formula for focal length of the electrostatic aperture (eV being kinetic energy of the beam; assume it does not change appreciably as the beam traverses the aperture):

$$f = 4V \frac{1+\frac{1}{2}eV/mc^2}{1+eV/mc^2}\frac{1}{E_2 - E_1}.$$

Solution:

From Gauss law one can show that $E_r = -\frac{r}{2}E_z'$ for small r. From the equation of motion, $\gamma m \frac{dv_r}{dt} = eE_r$, one finds $\gamma m \frac{dv_r}{dt} = \int eE_r dt$, or invoking $dt = dz/\beta c$ and $E_r = -\frac{r}{2}E_z'$ one arrives at (the radial velocity before the aperture is $v_r = 0$):

$$\gamma m \frac{dv_r}{dt} = -\frac{er}{2\beta c}\int E_z' dz = -\frac{er}{2\beta c}(E_2 - E_1).$$

The deflecting angle is $\alpha = v_r / v_z << 1$, and the focal length is $f = -r/\alpha$ (negative sign means defocusing). Thus, we arrive at

$$f = \frac{2v_z^2 \gamma m}{e(E_2 - E_1)} = \frac{2\beta^2 \gamma mc^2}{e(E_2 - E_1)}.$$

Using $\beta^2\gamma = \dfrac{\frac{eV}{mc^2}\left(2+\frac{eV}{mc^2}\right)}{1+\frac{eV}{mc^2}}$, we arrive at

$$f = 4V \frac{1+\frac{1}{2}eV/mc^2}{1+eV/mc^2}\frac{1}{E_2 - E_1}.$$