Fourier Optics

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Outline

- 2D Fourier Transform
- 4-f System
- Examples of spatial frequency filters
- Phase contrast imaging
- Matlab FFT

Properties of 2D Fourier Transforms

Definitions:

$$F(f_x, f_y) = \iint f(x, y) e^{-i2\pi(xf_x + yf_y)} dx dy$$

$$f(x, y) = \iint F(f_x, f_y) e^{i2\pi(xf_x + yf_y)} df_x df_y$$

Linearity:

$$\alpha f(x, y) + \beta g(x, y) \longleftrightarrow \alpha F(f_x, f_y) + \beta G(f_x, f_y)$$

Scaling:

$$f\left(\frac{x}{a}, \frac{y}{b}\right) \longleftrightarrow |ab| F(af_x, bf_y)$$

Shift:

$$f(x - x_0, y - y_0) \longleftrightarrow F(f_x, f_y) e^{-i2\pi(x_0 f_x + y_0 f_y)}$$

Properties of 2D Fourier Transforms (contd.)

Rotation:

$$R_\theta\left\{f(x,y)\right\} \longleftrightarrow R_\theta\left\{F(f_x,f_y)\right\}$$

Convolution:

$$\iint f(\tilde{x},\tilde{y})g(x-\tilde{x},y-\tilde{y})d\tilde{x}d\tilde{y} \longleftrightarrow F(f_x,f_y)G(f_x,f_y)$$

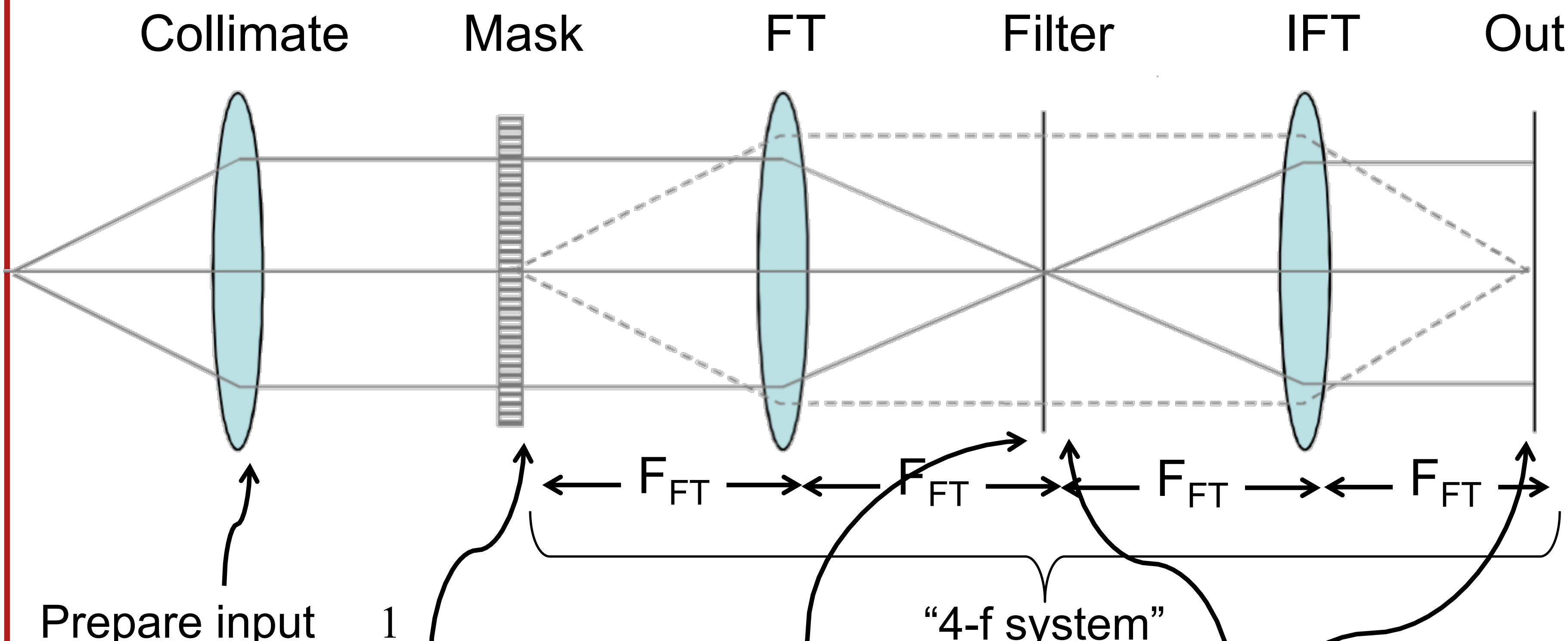
Parseval's theorem:

$$\iint |f(x,y)|^2 dxdy = \iint |F(f_x,f_y)|^2 df_x df_y$$

Slice theorem:

$$\int f(x,y)dy \longleftrightarrow F(f_x,0)$$

4-f System



Prepare input 1

Input mask $f(x,y)$

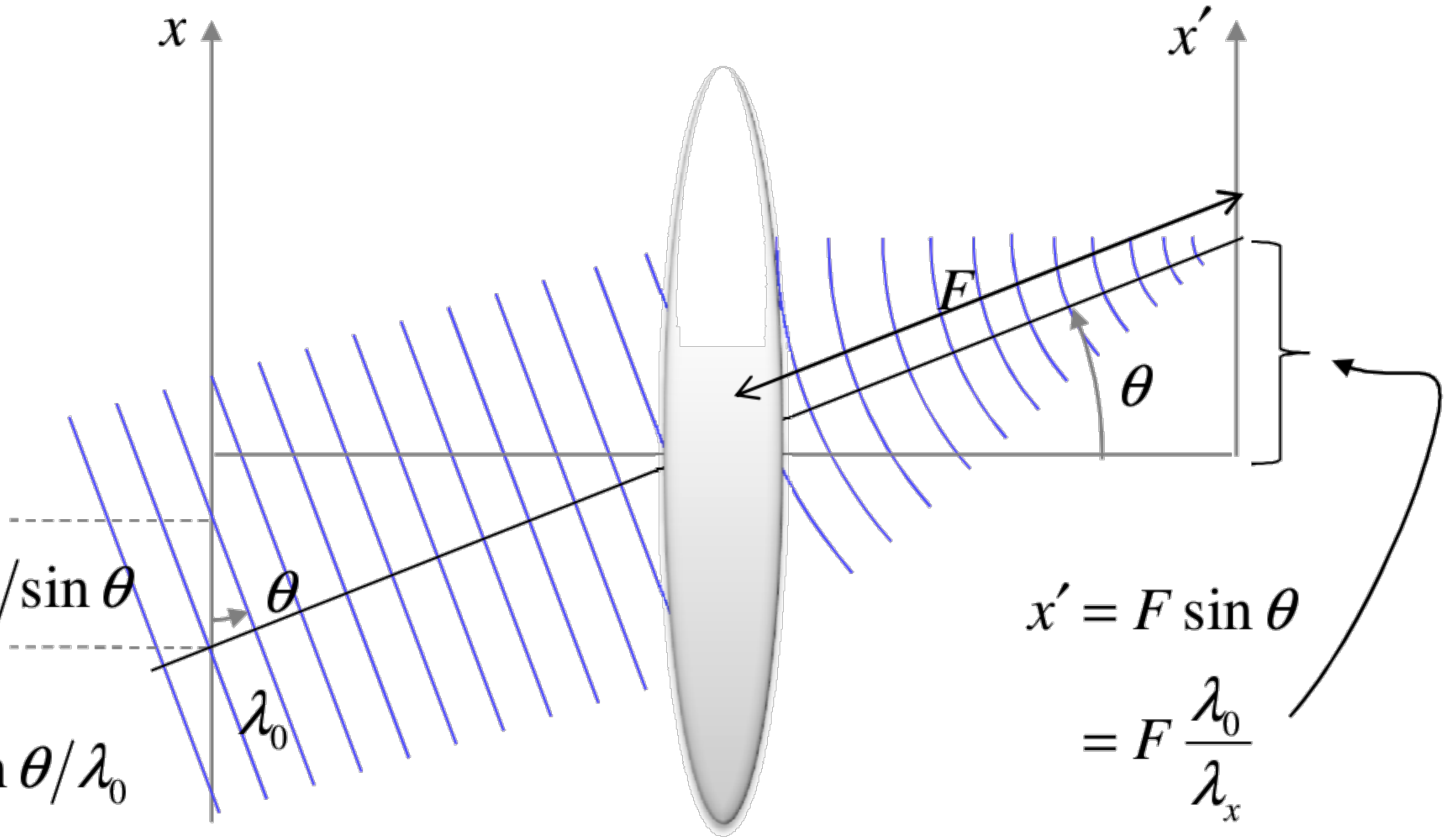
Take FT $F\left(\frac{x'}{\lambda F_{FT}}, \frac{y'}{\lambda F_{FT}}\right)$

Filter mask $F\left(\frac{x'}{\lambda F_{FT}}, \frac{y'}{\lambda F_{FT}}\right) G\left(\frac{x'}{\lambda F_{FT}}, \frac{y'}{\lambda F_{FT}}\right)$

Inverse FT $\iint f(\tilde{x},\tilde{y}) g(x-\tilde{x}, y-\tilde{y}) d\tilde{x} d\tilde{y}$

2D object f(x,y) has been filtered with 2D filter with impulse response g(x,y)

Coordinate system after the lens: spatial frequency converter



$$\lambda_x = \lambda_0 / \sin\theta$$

or

$$f_x = \sin\theta / \lambda_0$$

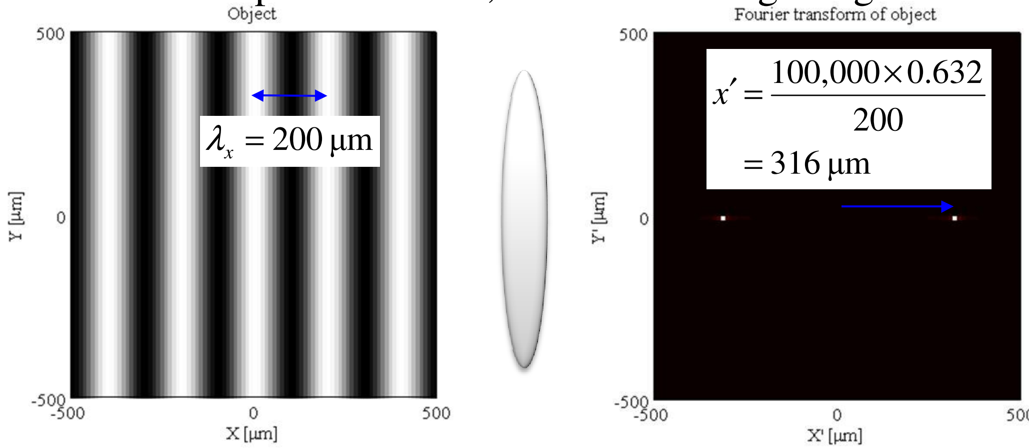
$$x' = F\sin\theta = F\frac{\lambda_0}{\lambda_x} = F\lambda_0 f_x$$

Thus, the spatial frequency f_x is related to coordinate x' by scaling factor Fλ

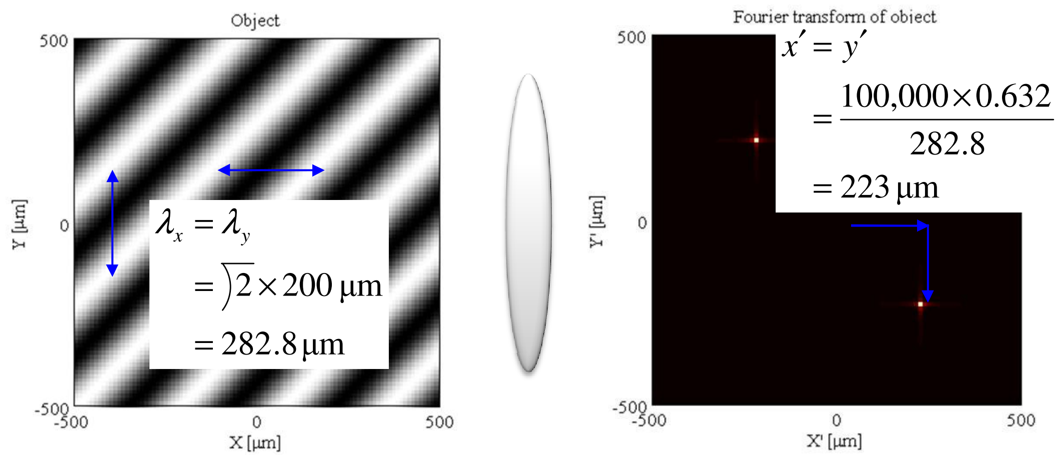
Simple Fourier Transform

Focal length F = 100 mm
Laser wavelength $\lambda_0 = 632$ nm

Amplitude cosine, aka diffraction grating

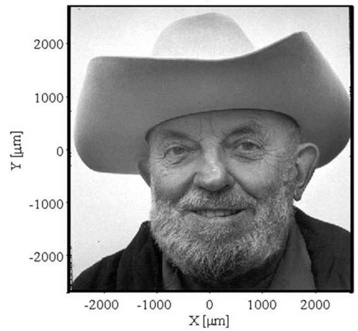


Rotate object by 45°

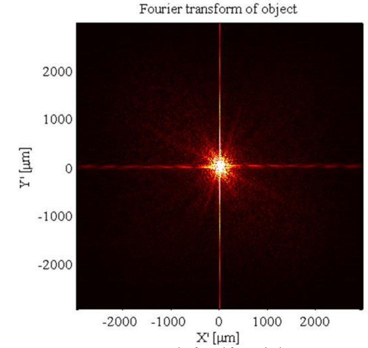


Low pass filter (sharp cutoff)

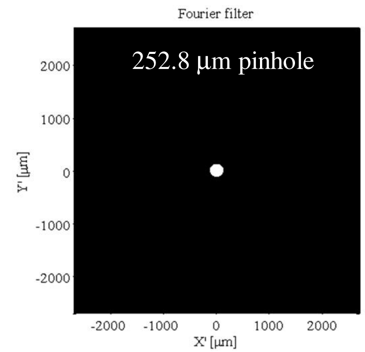
REAL SPACE



FOURIER SPACE



Multiplied by



Filter cutoff frequency = $1/500\ \mu m^{-1}$
Filter cutoff position = 126.4 µm
Focal length = 100 mm
Laser wavelength = 632 nm

All plots show amplitude of E

Smoothed, but Gibbs ringing due to sharp filter edges

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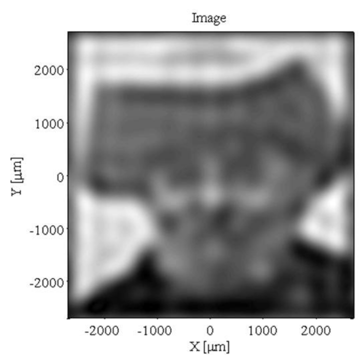
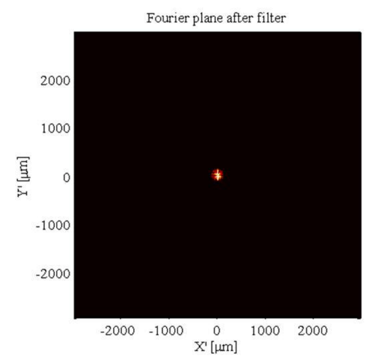
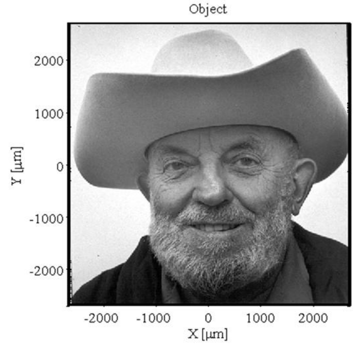




Low pass filter (smooth cutoff)

REAL SPACE



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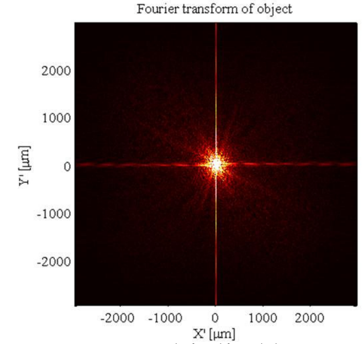
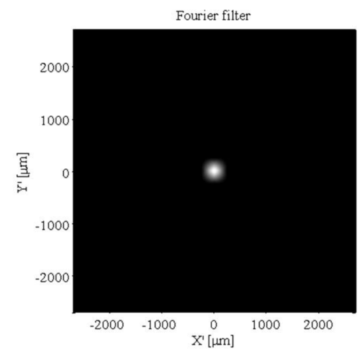
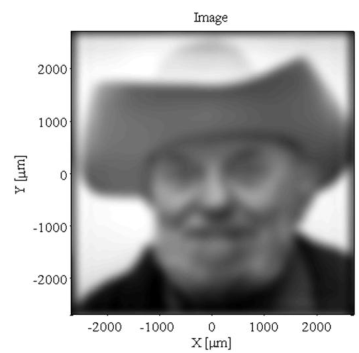


Multiplied by



Filter cutoff frequency = 1/500 μm^{-1}
Filter cutoff position = 126.4 µm
Edge smoothing = 132 µm
Focal length = 100 mm
Laser wavelength = 632 nm

Now just nicely smoothed



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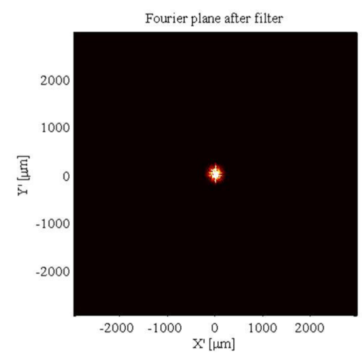
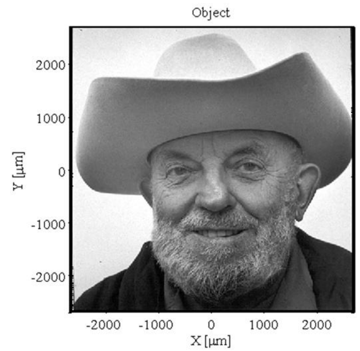
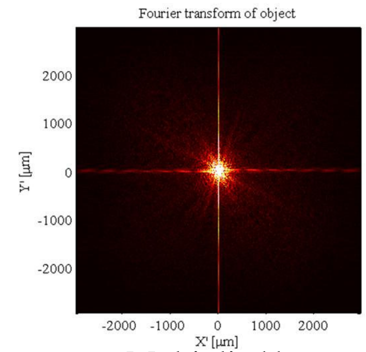


High pass (narrow band)

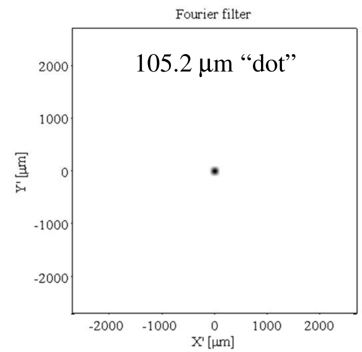
REAL SPACE



FOURIER SPACE



Multiplied by



Filter cutoff frequency	= 1/1200 μm^{-1}
Filter cutoff position	= 52.6 µm
Edge smoothing	= 58.2 µm
Focal length	= 100 mm
Laser wavelength	= 632 nm

Note sharp edges, darkening of large, uniform areas (~DC)

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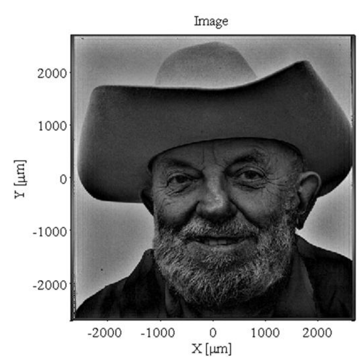
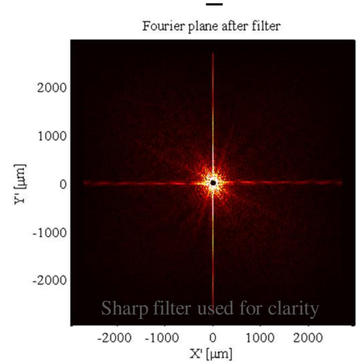
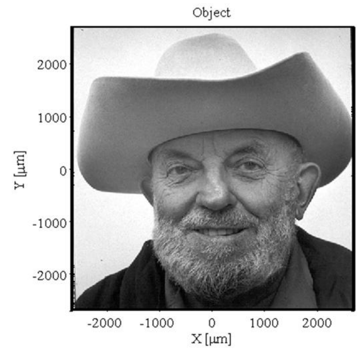




High pass (high band)

REAL SPACE



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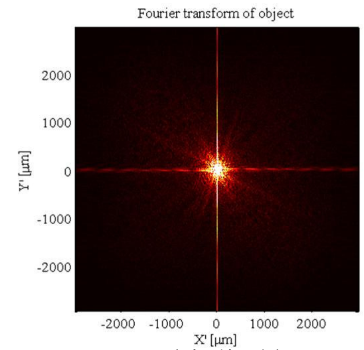

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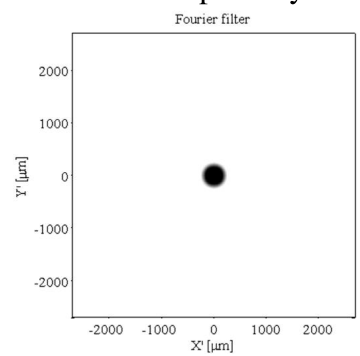

Filter cutoff frequency	= 1/300 μm^{-1}
Filter cutoff position	= 210.6 µm
Edge smoothing	= 46.6 µm
Focal length	= 100 mm
Laser wavelength	= 632 nm

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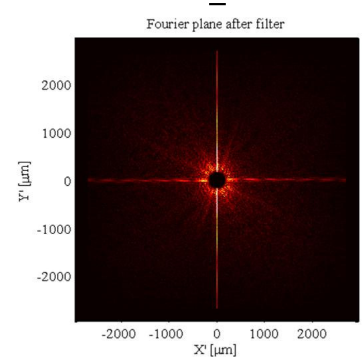
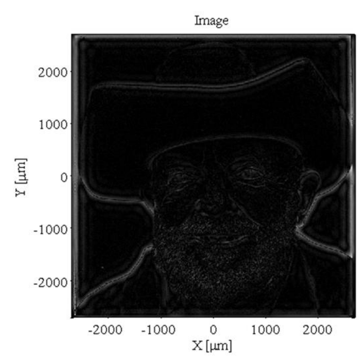
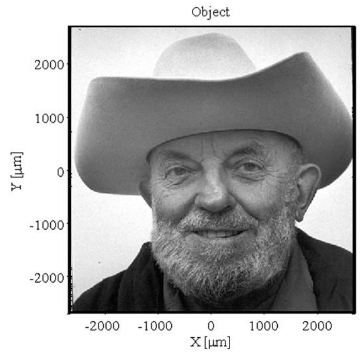


Only edges remain. Almost a "line drawing"

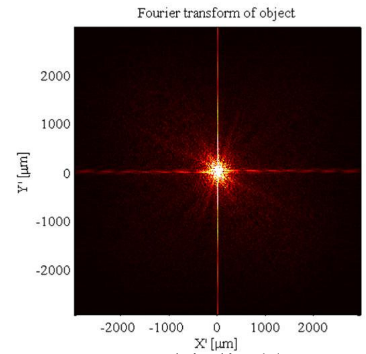


Vertical low pass ("smeers" vertically)

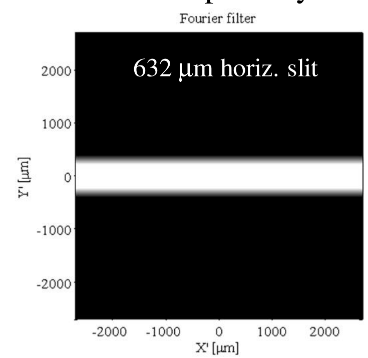
REAL SPACE



FOURIER SPACE



Multiplied by



Filter cutoff frequency	= 1/200 μm^{-1}
Filter cutoff position	= 316 µm
Edge smoothing	= 93.6 µm
Focal length	= 100 mm
Laser wavelength	= 632 nm

Horizontal lines at edges of eyes gone
Vertical lines above nose remain.

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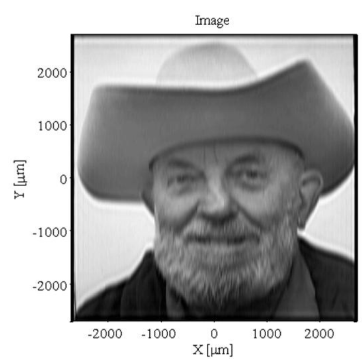
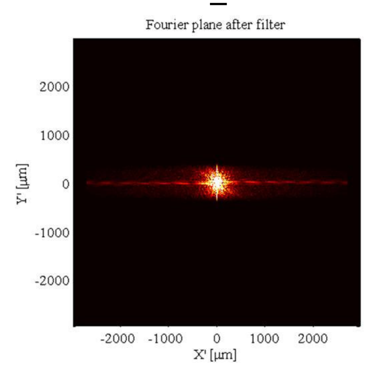
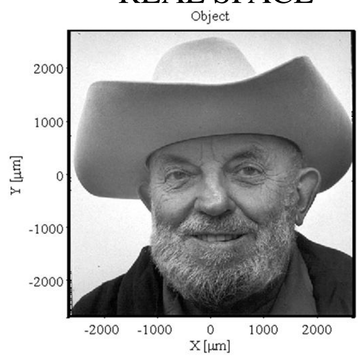


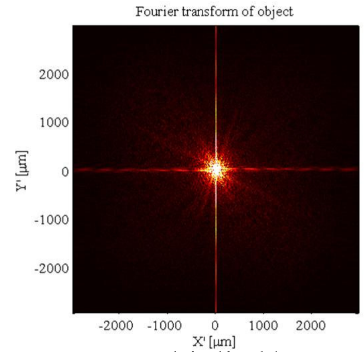


Horizontal low pass ("smeers" horizontally)

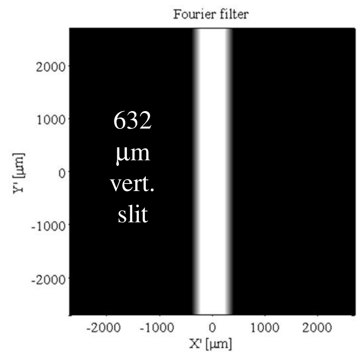
REAL SPACE



FOURIER SPACE



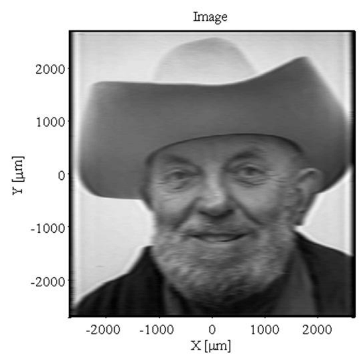
Multiplied by

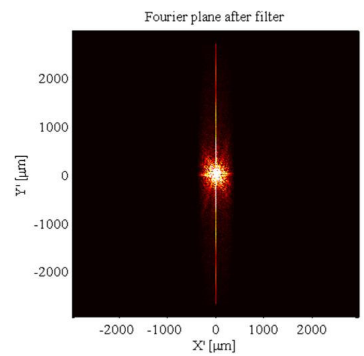


Filter cutoff frequency	= 1/200 μm^{-1}
Filter cutoff position	= 316 µm
Edge smoothing	= 93.6 µm
Focal length	= 100 mm
Laser wavelength	= 632 nm

Horizontal lines at edges of eyes remain.
Vertical lines above nose gone.

=

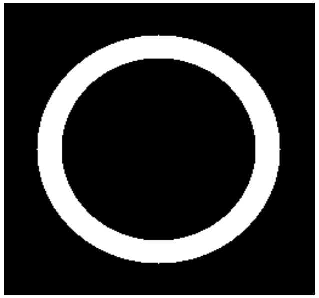




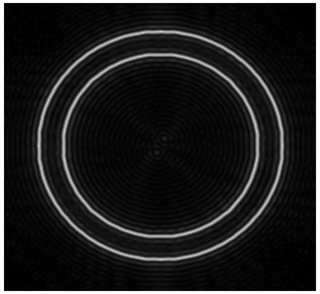
Simpler objects

Low-pass

Original



High-pass

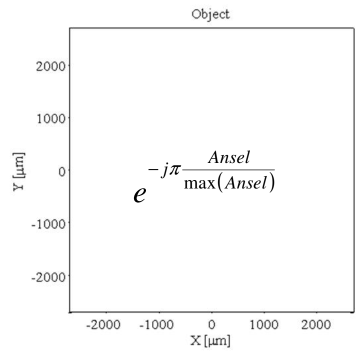


A side note: this is how bandpass filters look like in the frequency (focus) domain.

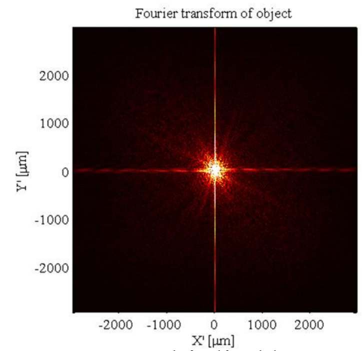
Low-pass, different cutoff in x&y

Phase contrast imaging

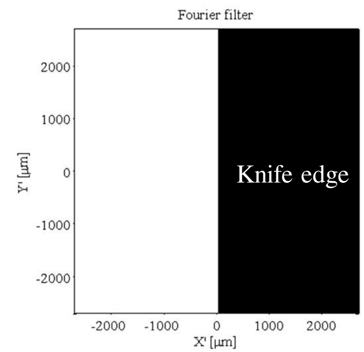
REAL SPACE



FOURIER SPACE

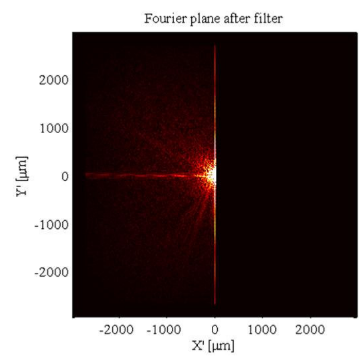


Multiplied by

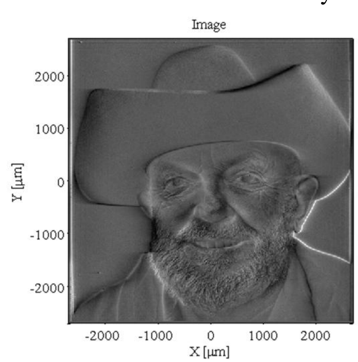


Filter cutoff frequency	$= 1/5000\ \mu m^{-1}$
Filter cutoff position	= 12.6 µm
Focal length	= 100 mm
Laser wavelength	= 632 nm

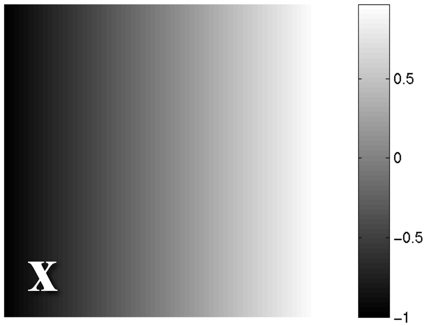
=



Phase has become amplitude.
Zernike won the 1953 Nobel in Physics for this.

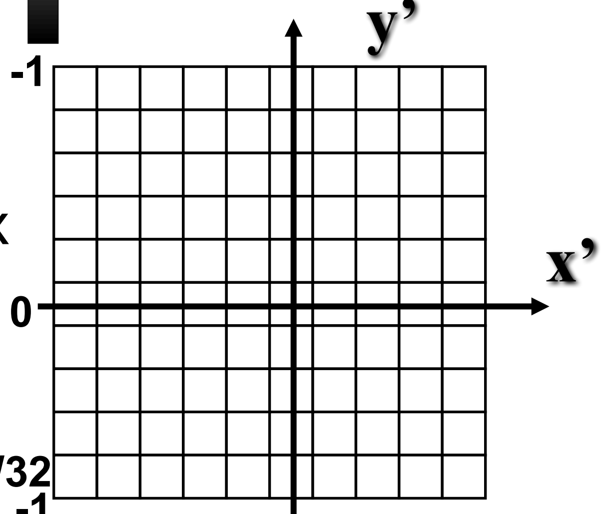


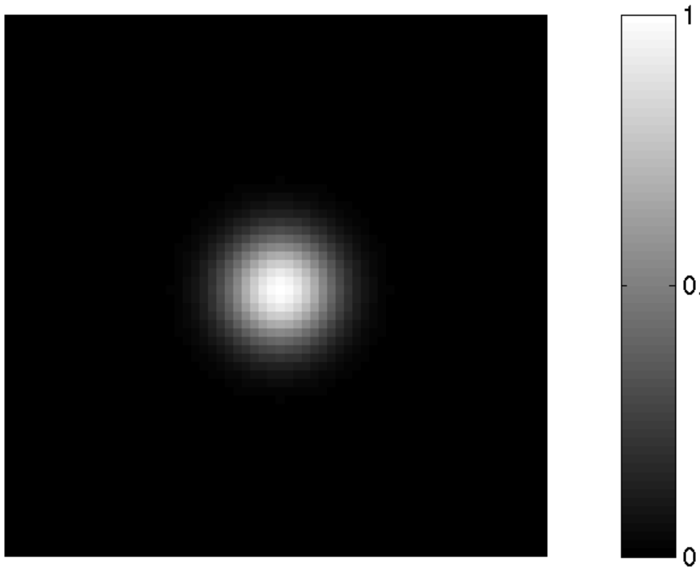
Some MATLAB tips: 2D functions



- **Create a matrix that evaluates 2D Gaussian: $\exp(-\pi/2(x^2+y^2)/\sigma^2)$**
  - **>>ind = [-32:1:31] / 32;**
  - **>>[x,y] = meshgrid(ind,-1*ind);**
  - **>>z = exp(-pi/2*(x.^2+y.^2)/(.25.^2));**
  - **>>imshow(z)**
  - **>>colorbar**

64x64 matrix

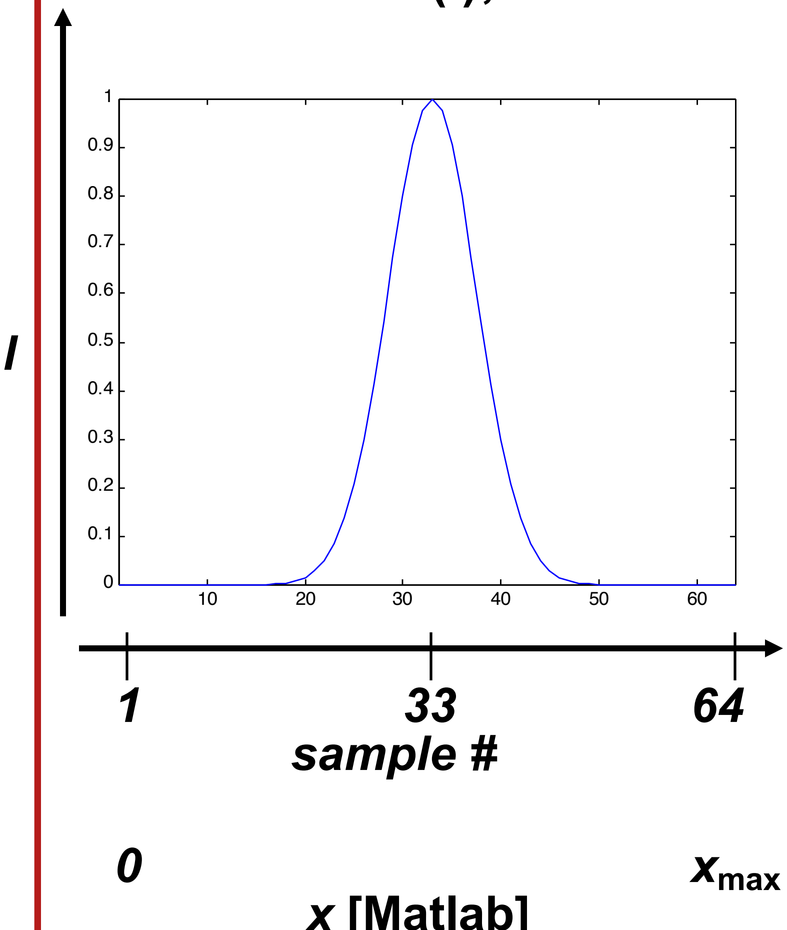




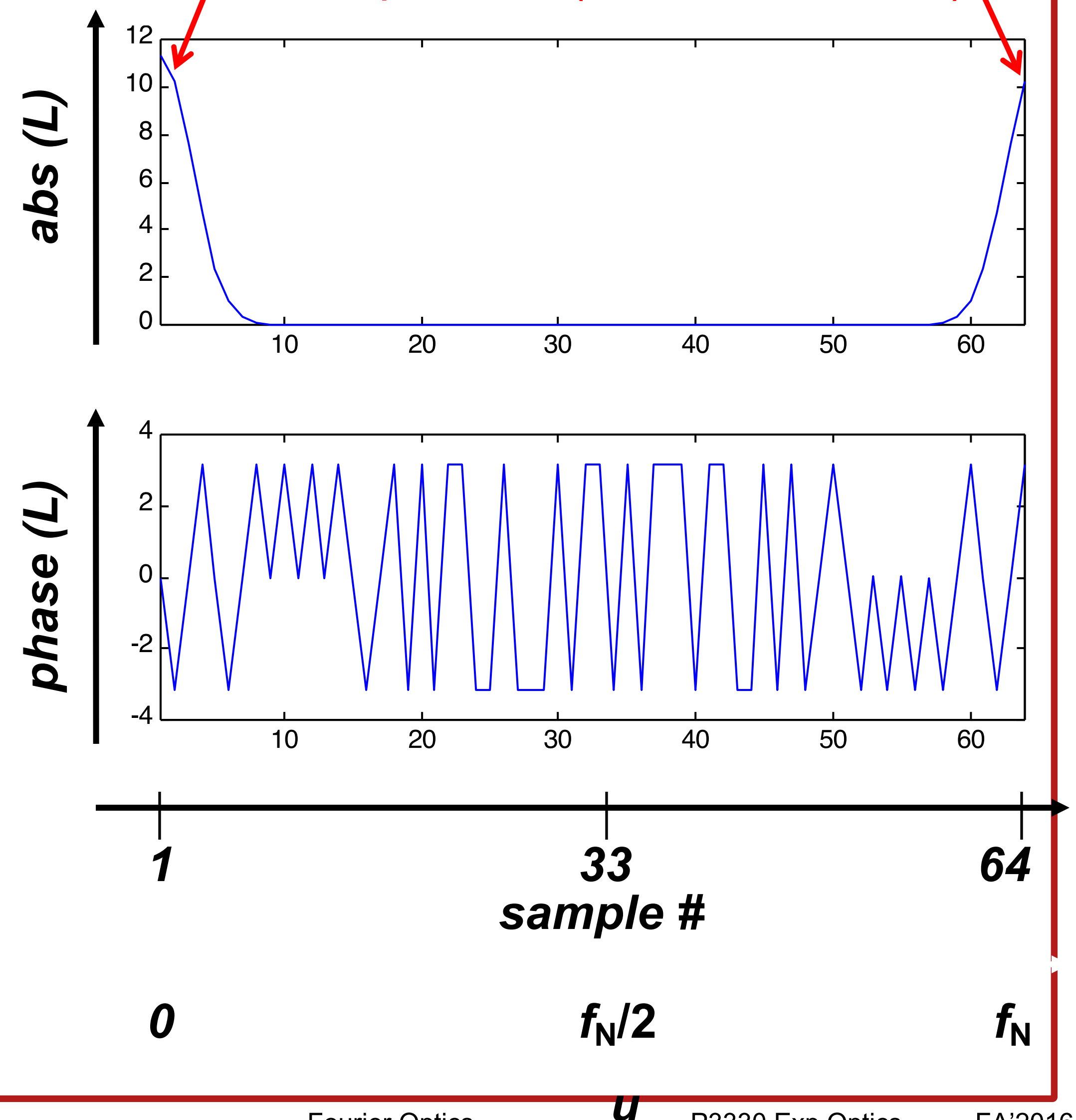
FFT "splits" low frequencies

1D Fourier Transform

- >>I = z(33,:);
- >>L = fft(I);



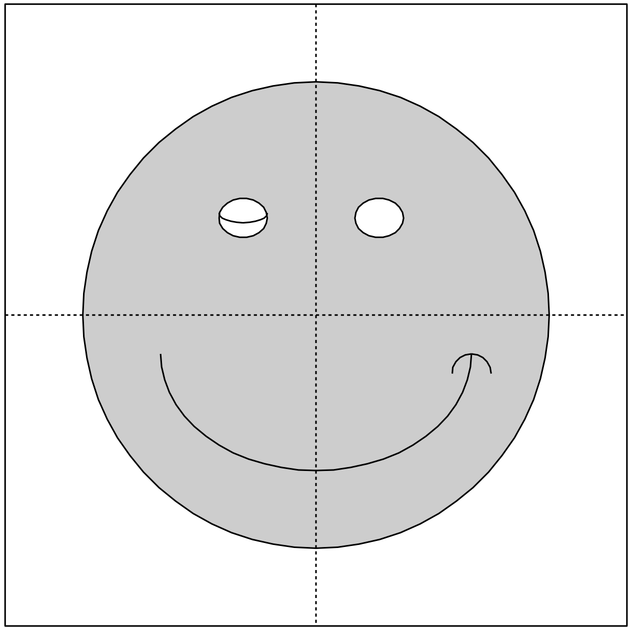
low frequencies (aka shift theorem)

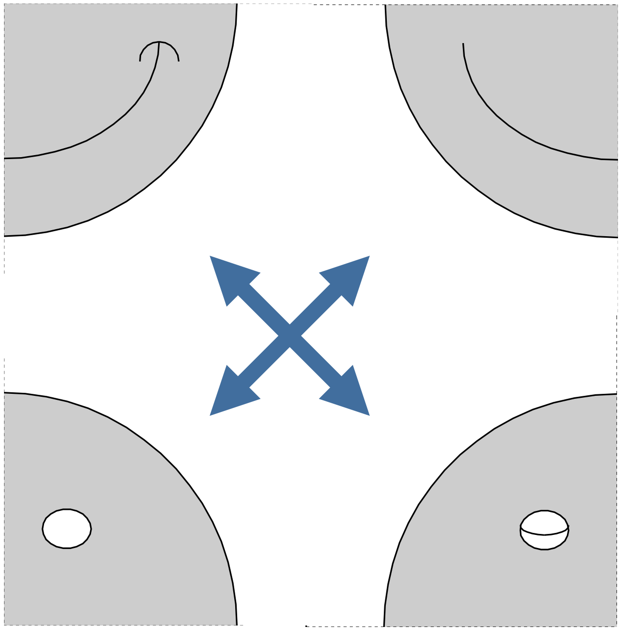


Use FFTSHIFT prior to/after FFT or FFT2

Use fftshift for 2D functions

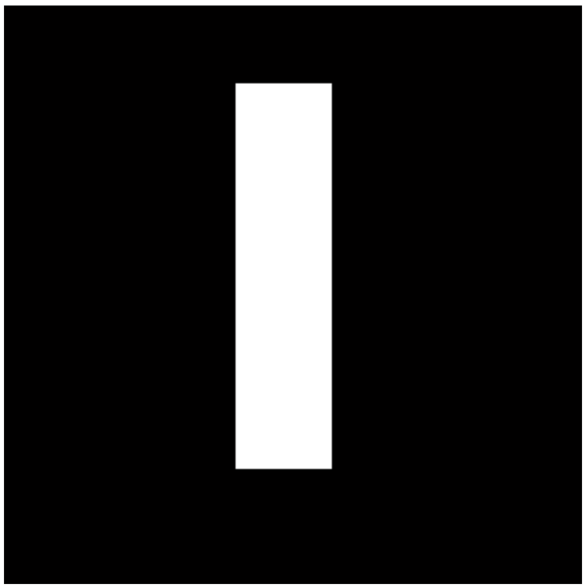
– >>smiley2 = fftshift(smiley);





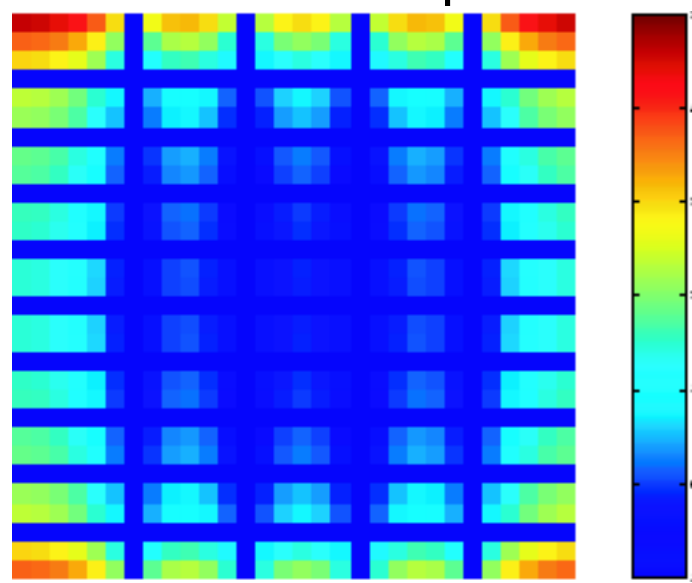
If your FFT looks jagged…

```
f = zeros(30,30);
f(5:24,13:17) = 1;
imshow(f,'InitialMagnification','fit')
```



```
F = fft2(f);
F2 = log(abs(F));
imshow(F2,[-1 5],'InitialMagnification','fit');
colormap(jet); colorbar
```

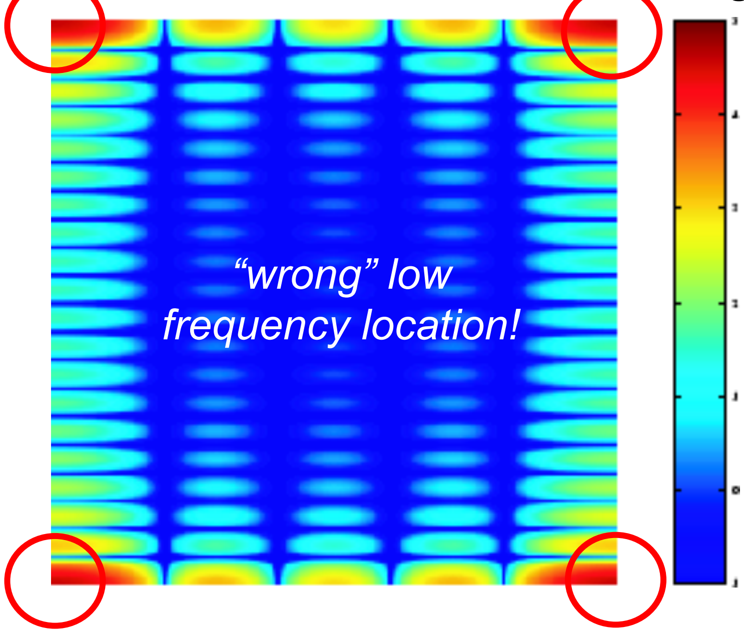
Discrete Fourier Transform Computed Without Padding



Apply zero-padding!

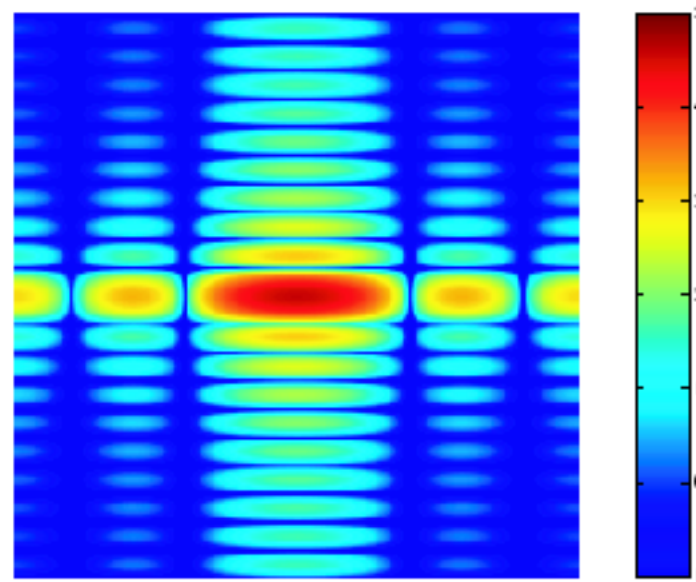
```
F = fft2(f,256,256);
imshow(log(abs(F)),[-1 5]); colormap(jet);
colorbar
```

Discrete Fourier Transform With Padding




```
% Apply FFTSHIFT
F = fft2(f,256,256);F2 = fftshift(F);
imshow(log(abs(F2)),[-1 5]); colormap(jet);
colorbar
```

Normal FT look



Links/references

http://ecee.colorado.edu/~mcleod/teaching/ugol/lecturenotes/Lecture%204%20Fourier%20Optics.pdf

http://www.medphysics.wisc.edu/~block/bme530lectures/matlabintro.ppt

http://www.mathworks.com/help/images/fourier-transform.html