Transverse Quadrupole BBU Threshold Current in the Cornell X-ray ERL

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Abstract

The total current that can be accelerated in an Energy Recovery Linear Accelerator (ERL) can be limited by the longitudinal monopole beam breakup (BBU) instability, the transverse dipole BBU instability and the transverse quadrupole BBU instability. The quadrupole HOMs can be induced by a beam with non-zero quadrupole moment and they can in turn, alter the beta function and phase advance by providing a focusing effect. In this paper we study the how the quadrupole BBU threshold current for an ERL can be evaluated and its impact on the cavity design. An analytic formula is derived for the simple case with a single mode that is oriented in x/y direction in one cavity. The threshold current for the realistic ERL lattice is computed by simulation. The result from simulation is also compared with the analytic formula.

1 Introduction

2 Theoretical Analysis of the Quadrupole Beam Breakup Instability

2.1 HOM Voltage Induced by a single Bunch

In an RF cavity, the total HOM energy stored can be written as

$$U = \alpha V^2(r_0, \varphi_0) \tag{1}$$

where $V(r_0, \varphi_0)$ is the amplitude of the HOM voltage measured at position $(r_0,\ \varphi_0)$, and α is a constant determined by the geometry of the cavity.

Travelling through the cavity at position $(r,\ \varphi)$, a bunch with charge dq can induce more HOM voltage and add energy to the existing HOM:

$$U + dU = \alpha V^2(r_0, \varphi_0) + dqV(r, \varphi) = \alpha \tilde{V}^2(r_0, \varphi_0) \tag{2}$$

where $\tilde{V}^2(r_0, \varphi_0)$ is the new HOM voltage at position $(r_0,\ \varphi_0)$ after the charged bunch traversed the RF cavity. Thus the new HOM voltage can be calculated as

$$\tilde{V}(r_0, \varphi_0) = \sqrt{V^2(r_0, \varphi_0) + \frac{dq}{\alpha} V(r, \varphi)} \approx V(r_0, \varphi_0) + \frac{dq}{2\alpha} \frac{V(r, \varphi)}{V(r_0, \varphi_0)} \ , \tag{3}$$

and the change in the HOM voltage is

$$dV(r_0, \varphi_0) = \frac{dq}{2\alpha} \frac{V(r, \varphi)}{V(r_0, \varphi_0)} \ . \tag{4}$$

Define $K(r_0, \varphi_0)$ as the loss factor for a charged particle traversing the RF cavity at position $(r_0,\ \varphi_0)$, we have

$$K(r_0, \varphi_0) = \frac{|V(r_0, \varphi_0)|^2}{4U} = \frac{1}{4\alpha}. \tag{5}$$

Thus the change in the HOM voltage can be written as

$$dV(r_0, \varphi_0) = 2dqK(r_0, \varphi_0)\frac{V(r, \varphi)}{V(r_0, \varphi_0)} \ , \tag{6}$$

and the total HOM voltage change induced by the whole charged bunch is

$$V(r_0, \varphi_0) = 2qK(r_0, \varphi_0)\int \frac{V(r, \varphi)}{V(r_0, \varphi_0)}\rho(r, \varphi)rdrd\varphi \ , \tag{7}$$

where q is the total charge of the bunch and $\rho(r, \varphi)$ is the charge density function.

2.2 Quadrupole HOM Voltage

The voltage of a quadrupole HOM can be written as

$$V(r, \varphi) = V_0 r^2 \cos 2(\varphi - \phi_0) \tag{8}$$

and Eq. 7 becomes

$$V(r_0, \varphi_0) = \frac{2qK(r_0, \varphi_0)}{r_0^2 \cos 2(\varphi_0 - \phi_0)}\int r^2 \cos 2(\varphi - \phi_0)\rho(r, \varphi)rdrd\varphi \ , \tag{9}$$

With $x = r\cos\varphi$ and $y = r\sin\varphi$, we have

$$r^2 \cos 2(\varphi - \phi_0) = (x^2 - y^2)\cos 2\phi_0 + 2xy \sin 2\phi_0 \tag{10}$$

and Eq. 9 becomes

$$V(r_0, \varphi_0) = \frac{2qK(r_0, \varphi_0)}{r_0^2 \cos 2(\varphi_0 - \phi_0)}\int \left\{(x^2 - y^2)\cos 2\phi_0 + 2xy \sin 2\phi_0\right\}\rho(x, y)dxdy \ , \tag{11}$$

Define the normal and skewed quadrupole moment of the beam as

$$\begin{aligned} Q_n &= \int (x^2 - y^2)\rho(x, y)dxdy = \langle x^2 \rangle \\ Q_s &= \int 2xy\rho(x, y)dxdy = 2\langle xy \rangle \end{aligned} \tag{12}$$

we can find out the quadrupole HOM voltage induced by a beam with Q_n and Q_s, at $(r_0,\ \varphi_0)$ and t_0:

$$V(r_0, \varphi_0) = \frac{2qK(r_0, \varphi_0)}{r_0^2 \cos 2(\varphi_0 - \phi_0)}\left\{Q_n \cos 2\phi_0 + Q_s \sin 2\phi_0\right\} \ . \tag{13}$$

Thus the quadrupole HOM voltage excited by a bunch and observed at r, φ and t can be written as

$$V(r, \varphi, t) = V(r_0, \varphi_0)\left(\frac{r}{r_0}\right)^2 \frac{\cos 2(\varphi - \phi_0)}{\cos 2(\varphi_0 - \phi_0)} e^{-\frac{\omega_\lambda}{2Q_\lambda}(t - t_0)} \cos\omega_\lambda(t - t_0) \ . \tag{14}$$

where $V(r_0, \varphi_0)$ is the maximum HOM voltage at $(r_0,\ \varphi_0)$.

2.3 Quadrupole Wake Fields

The quadrupole wake function is defined as the work done by the quadrupole HOM on a test charge e:

$$W_z(x, y, s) = -eV(r, \varphi, t)|_{t=s/c} \ . \tag{15}$$

We can also write the longitudinal wake function in terms of the HOM voltage at r, φ and t_0, which gives

$$\begin{aligned} W_z(x, y, t) &= -eV(r_0, \varphi_0)\left(\frac{r}{r_0}\right)^2 \frac{\cos 2(\varphi - \phi_0)}{\cos 2(\varphi_0 - \phi_0)} e^{-\frac{\omega_\lambda}{2Q_\lambda}t} \cos\omega_\lambda t \\ &= -\frac{eV(r_0, \varphi_0)}{r_0^2 \cos 2(\varphi_0 - \phi_0)}\left\{(x^2 - y^2)\cos 2\phi_0 + 2xy \sin 2\phi_0\right\} e^{-\frac{\omega_\lambda}{2Q_\lambda}t} \cos\omega_\lambda t \end{aligned} \tag{16}$$

In order to find the transverse wake function, we can write the longitudinal wake function as the longitudinal gradient of a scalar potential Ψ:

$$W_z = \partial_s \Psi \approx \frac{1}{c}\partial_t \Psi \; . \tag{17}$$

Thus Ψ can be calculated as

$$\begin{aligned}
\Psi &= c\int dt W_z(x,y,t) \\
&= -\frac{ecV(r_0,\varphi_0)}{r_0^2\cos 2(\varphi_0-\phi_0)}\left\{(x^2-y^2)\cos 2\phi_0 + 2xy\sin 2\phi_0\right\}\int dt e^{-\frac{\omega_\lambda}{2Q_\lambda}t}\cos\omega_\lambda t \\
&= -e\frac{c}{\omega_\lambda}\frac{V(r_0,\varphi_0)}{r_0^2\cos 2(\varphi_0-\phi_0)}\left\{(x^2-y^2)\cos 2\phi_0 + 2xy\sin 2\phi_0\right\}e^{-\frac{\omega_\lambda}{2Q_\lambda}t}\sin\omega_\lambda t
\end{aligned} \tag{18}$$

The transverse wake function can be calculated from Ψ

$$\begin{aligned}
W_x &= \partial_x\Psi = -e\frac{c}{\omega_\lambda}\frac{2V(r_0,\varphi_0)}{r_0^2\cos 2(\varphi_0-\phi_0)}\left\{x\cos 2\phi_0 + y\sin 2\phi_0\right\}e^{-\frac{\omega_\lambda}{2Q_\lambda}t}\sin\omega_\lambda t \\
W_y &= \partial_y\Psi = -e\frac{c}{\omega_\lambda}\frac{2V(r_0,\varphi_0)}{r_0^2\cos 2(\varphi_0-\phi_0)}\left\{-y\cos 2\phi_0 + x\sin 2\phi_0\right\}e^{-\frac{\omega_\lambda}{2Q_\lambda}t}\sin\omega_\lambda t
\end{aligned} \tag{19}$$

and the transverse force $\vec{F}$ can be written as

$$\vec{F} = \frac{1}{L}\vec{W} \tag{20}$$

From Eq. 13 we can write the transverse forces in terms of the loss factor

$$\begin{aligned}
F_x &= -\frac{4eq}{L}\frac{c}{\omega_\lambda}\frac{K(r_0,\varphi_0)}{r_0^4\cos^2 2(\varphi_0-\phi_0)}\left\{Q_n x\cos^2 2\phi_0 + Q_s y\sin^2 2\phi_0 + (Q_n y + Q_s x)\cos 2\phi_0\sin 2\phi_0\right\}e^{-\frac{\omega_\lambda}{2Q_\lambda}t}\sin\omega_\lambda t \\
F_y &= -\frac{4eq}{L}\frac{c}{\omega_\lambda}\frac{K(r_0,\varphi_0)}{r_0^4\cos^2 2(\varphi_0-\phi_0)}\left\{-Q_n y\cos^2 2\phi_0 + Q_s x\sin^2 2\phi_0 + (Q_n x - Q_s y)\cos 2\phi_0\sin 2\phi_0\right\}e^{-\frac{\omega_\lambda}{2Q_\lambda}t}\sin\omega_\lambda t
\end{aligned} \tag{21}$$

If we choose the loss factor measured at $\varphi_0 = \phi_0$, we have

$$K(r_0,\phi_0) = \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \tag{22}$$

Define the quadrupole wake function as

$$W(t) = \frac{c}{2}\left(\frac{R}{Q}\right)_\lambda \frac{1}{r_0^4}e^{-\frac{\omega_\lambda}{2Q_\lambda}t}\sin\omega_\lambda t \tag{23}$$

and the transverse force can be written as

$$\begin{aligned}
F_x &= -\frac{4eq}{L}W(t)\left\{Q_n x\cos^2 2\phi_0 + Q_s y\sin^2 2\phi_0 + (Q_n y + Q_s x)\cos 2\phi_0\sin 2\phi_0\right\} \\
F_y &= -\frac{4eq}{L}W(t)\left\{-Q_n y\cos^2 2\phi_0 + Q_s x\sin^2 2\phi_0 + (Q_n x - Q_s y)\cos 2\phi_0\sin 2\phi_0\right\}
\end{aligned} \tag{24}$$

The orientation of the quadrupole mode ϕ_0 depends on the geometry of the RF cavity. For simplicity reasons, we only consider an upright quadrupole mode in our calculation ($\phi_0 = 0$). The transverse force measured $\vec{F}(t)$ is thus written as

$$\vec{F}(t) = -\frac{4eq}{L}\left\{x\vec{e}_x - y\vec{e}_y\right\}W(t-t')Q_n(t') \tag{25}$$

2.4 Quadrupole Beam Breakup Instability

Equation 25 shows the quadrupole force at t from the quadrupole mode excited by a beam at t_0. The formula can be extended for the CW beam case, in which the quadrupole mode is excited by all traversing bunches prior to t. For simplicity reasons, we consider a beam with normal quadrupole moment only. For a CW beam, its quadrupole moment is a function of time $Q_n = Q_n(t')$ and the force, as a result, can be written as:

$$\vec{F}(t) = -\frac{4eq}{L}\int_{-\infty}^{t}\{x\vec{e}_x - y\vec{e}_y\}W(t-t')Q_n(t')dt' \tag{26}$$

Define the quadrupole wake potential $G(t)$ as

$$G(t) = \int_{-\infty}^{t} W(t-t')Q(t')dt' \tag{27}$$

and thus the force becomes

$$\vec{F}(t) = -\frac{4eq}{L}G(t)\{x\vec{e}_x - y\vec{e}_y\} \tag{28}$$

The equation of motion for a test charge can be written as

$$x'' = -\frac{4eqG(t)}{\gamma m_e c^2 L}x = -kx \tag{29}$$

where we have

$$kL = \frac{4eq}{\gamma m_e c^2}G(t) \tag{30}$$

For an ERL with a single RF cavity the change of the beta function at t as a result of a quadrupole kick at $t - t_r$ can be written as

$$\begin{aligned}\Delta\beta_{x2}(t) &= -kL\beta_{x1}\beta_{x2}\sin 2(\psi_{x2}-\psi_{x1}) = -\frac{4eq}{\gamma m_e c^2}G(t-t_r)\beta_{x1}\beta_{x2}\sin 2(\psi_{x2}-\psi_{x1})\\ \Delta\beta_{y2}(t) &= +kL\beta_{y1}\beta_{y2}\sin 2(\psi_{y2}-\psi_{y1}) = +\frac{4eq}{\gamma m_e c^2}G(t-t_r)\beta_{y1}\beta_{y2}\sin 2(\psi_{y2}-\psi_{y1})\end{aligned} \tag{31}$$

For a CW beam, its normal quadrupole moment can be written as

$$Q_n(t) = Q_n\sum_{m=-\infty}^{+\infty}\delta(t-t_r-mt_b) \tag{32}$$

where t_b is the bunching period and Q_n can be calculated as

$$Q_n = \langle x^2\rangle - \langle y^2\rangle = \frac{\varepsilon_n}{\gamma}(\beta_x - \beta_y) \tag{33}$$

where ε_n is the normalized emittance of the beam.

Assuming the designed beam is round ($\beta_x = \beta_y$), we have

$$Q_n(t) = \frac{\varepsilon_n}{\gamma}(\Delta\beta_x - \Delta\beta_y)\sum_{m=-\infty}^{+\infty}\delta(t-t_r-mt_b) \tag{34}$$

Combining Eq. 27 and Eq. 34, we can derive an integral equation of the wake potential $G(t)$ as

$$\begin{aligned}G(t) &= -\frac{4eq\varepsilon_n}{\gamma^2 m_e c^2}[\beta_{x1}\beta_{x2}\sin 2\Delta\psi_x + \beta_{y1}\beta_{y2}\sin 2\Delta\psi_y]\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{t}W(t-t')G(t'-t_r)\delta(t'-t_r-mt_b)dt'\\ &== -\frac{4eq\varepsilon_n}{\gamma^2 m_e c^2}[\beta_{x1}\beta_{x2}\sin 2\Delta\psi_x + \beta_{y1}\beta_{y2}\sin 2\Delta\psi_y]\sum_{m=-\infty}^{n}W(t-t_r-mt_b)G(mt_b)\end{aligned} \tag{35}$$

With $q = I_0 t_b$, we can derive the formula for I_0

$$I_0^{-1} = -\frac{4e\varepsilon_n t_b}{\gamma^2 m_e c^2}[\beta_{x1}\beta_{x2}\sin 2\Delta\psi_x + \beta_{y1}\beta_{y2}\sin 2\Delta\psi_y]e^{i\omega n_r t_b}w(\delta) \tag{36}$$

where

$$\begin{aligned} w(\delta) &= \sum_{n=0}^{\infty} W([n+\delta]t_b)e^{i\omega n t_b} \\ &= \frac{c}{4r_0^4}\left(\frac{R}{Q}\right)_\lambda e^{-i\omega\delta t_b}\frac{e^{i\omega^\dagger(\delta-1)t_b}\sin\omega_\lambda\delta t_b - e^{i\delta\omega^\dagger t_b}\sin\omega_\lambda[\delta-1]t_b}{\cos\omega^\dagger t_b - \cos\omega_\lambda t_b} \end{aligned} \tag{37}$$

where $\omega^\dagger t_b = \omega t_b + i\epsilon$, $\epsilon = \omega_\lambda t_b/2Q_\lambda$ and $t_r = (n-\delta)t_b$. For an ERL, we have $\delta = \frac{1}{2}$, and the current is

$$I_0 = -\frac{\gamma^2 m_e c r_0^4}{e\left(\frac{R}{Q}\right)_\lambda \varepsilon_n t_b}\frac{e^{-i\omega t_r}}{\beta_{x1}\beta_{x2}\sin 2\Delta\psi_x + \beta_{y1}\beta_{y2}\sin 2\Delta\psi_y}\frac{\cos\omega^\dagger t_b - \cos\omega_\lambda t_b}{\cos\frac{\omega^\dagger t_b}{2}\sin\frac{\omega_\lambda t_b}{2}} \tag{38}$$

The approximate threshold current with $\epsilon \ll 1$ and $n_r\epsilon \ll 1$ can be derived as

$$I_0 = -\frac{\omega_\lambda\gamma E_e}{2\frac{ec}{r_0^4}\left(\frac{R}{Q}\right)_\lambda Q_\lambda\varepsilon_n}\frac{1}{\beta_{x1}\beta_{x2}\sin 2\Delta\psi_x + \beta_{y1}\beta_{y2}\sin 2\Delta\psi_y}\frac{1}{\sin\omega_\lambda t_r} \tag{39}$$

where E_e is the beam energy, r_0 is the radius where $\left(\frac{R}{Q}\right)_\lambda$ is measured and Q_λ is the quality factor.

3 Simulation Results for Quadrupole BBU Threshold Current

3.1 A Single Cavity with One Quadrupole HOM

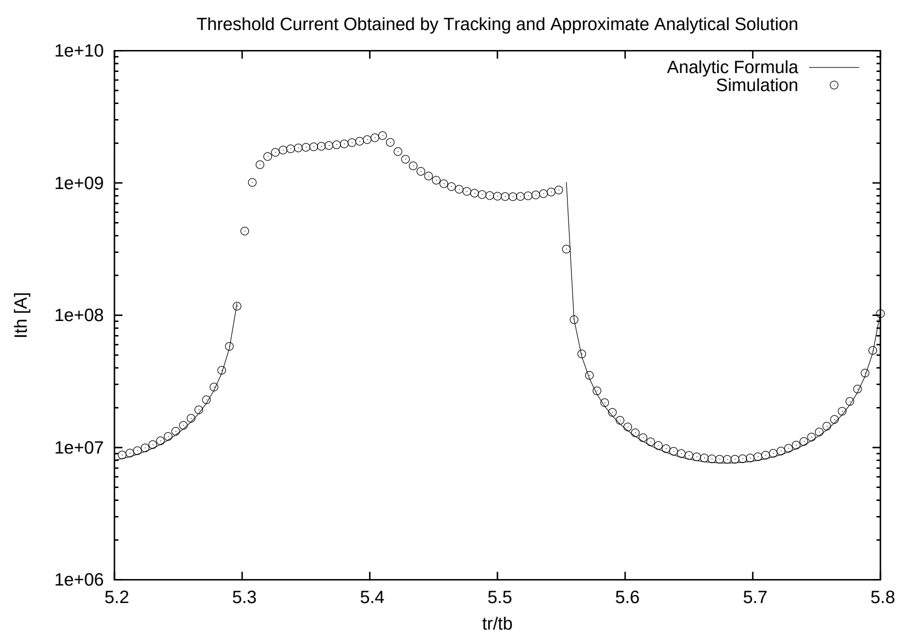

Figure 1: Threshold current obtained by tracking and by approximate analytical formula (Eq. 39. Parameters: $(R/Q)_\lambda = 34\Omega$, $Q_\lambda = 10^4$ and $\omega_\lambda = 2.575$GHz.

3.2 Quadrupole BBU Threshold Current of Cornell ERL

In an RF cavity for Cornell ERL, we have the iris radius $r_0 = 0.035$m and the Q_0 of the fundamental mode is about 10^{10}. The quadrupole BBU threshold current according to Eq. 39 with parameters listed in Tab. 1 is about 50**A**. This result is calculated for electrons with low energy. Because the threshold current is proportional to γ^2 of the beam, the estimate for the 5**GeV** ERL should be higher than 50**A**.

Table 1: Parameters for an RF cavity with a single quadrupole mode

$f_\lambda = 3.256$GHz	$(R/Q)_\lambda = 1.595\Omega$	$Q_\lambda = 2.59 \times 10^6$	$t_r = 6.8132 \times 10^{-6}$s
$\varphi_{x1} = 0.258631$rad	$\varphi_{y1} = 0.258631$rad	$\varphi_{x2} = 329.080512$rad	$\varphi_{y2} = 244.583540$rad
$\beta_{x1} = 17.65734$m	$\beta_{y1} = 17.65734$m	$\beta_{x2} = 83.154592$m	$\beta_{y2} = 70.6044042$m

The simulation for the full ERL lattice with the dominant modes (No.1, 2, 3, and 5 in Tab. 2) and a reduced $Q = Q_\lambda \cdot 10^{-3}$ gives a threshold current about 200**A**. If a 0.3% ($\approx$ 10**MHz**) frequency spread is introduced to the quadrupole HOMs, the threshold current can be increased to about 1200**A**. Reducing the Q value is necessary because the high value of Q requires extremely long time to damp the HOM significantly. In order to avoid excessive simulation time, we run the program for an artificially smaller Q and deduce the threshold current of the high Q according to Eq. 39. The simulation results for the single cavity case and the full ERL lattice case are listed in Tab. 3.

Table 2: The dominant quadrupole HOMs for the 7-cell ERL cavity.

	f_λ[GHz]	Q_λ/Q_0	$(R/Q)_\lambda[\Omega/\text{cm}^4]$	$(R/Q)_\lambda[\Omega]$	$(R/Q)_\lambda \cdot Q_\lambda[\Omega]$
1	2.3052	0.570	0.052267	0.96196	5.4832×10^9
2	2.3074	0.572	0.045267	0.82995	4.7473×10^9
3	2.4896	0.516	0.060044	0.81231	4.1915×10^9
4	3.2414	0.256	0.154078	0.72540	1.8570×10^9
5	3.2532	0.259	0.344944	1.60056	4.1454×10^9
6	3.2670	0.263	0.217078	0.99034	2.6046×10^9
7	3.4860	0.315	0.106633	0.37527	1.1821×10^9
8	3.5144	0.328	0.049389	0.16826	5.5190×10^8
9	3.8531	0.251	0.061756	0.14561	3.6549×10^8

Table 3: Simulation results of the quadrupole BBU threshold for the 7-cell ERL cavity.

	$Q_\lambda \cdot 10^{-3}$	Q_λ
Single Cavity at Low Energy	50**A**	50 **mA**
Full ERL Lattice without Frequency Spread	200**A**	200 **mA**
Full ERL Lattice with 10 MHz Frequency Spread	1200**A**	1.2**A**

4 Higher Order Multipole BBU Instability

$$dV(w_0, t, t') = \frac{2K_{\text{loss}}(w_0)}{g(w_0)} \int_\infty j(w, t') f_z(t, t') g(w) dw \tag{40}$$

$$dV(w_0, t, t') = dV(t, t') g(w_0) \tag{41}$$

$$V_\perp(t) = -\left(\frac{c}{\omega_\lambda}\right) \frac{2K_{\text{loss}}(w_0)}{g^2(w_0)} \int_{-\infty}^{t} \int_\infty j(w, t') f_\perp(t, t') g(w) dw dt' \tag{42}$$

$$j_0(w, t - t_r) = j_1(\vec{T}(w), t) \tag{43}$$

$$\begin{aligned} V_\perp(t) &= -\left(\frac{c}{\omega_\lambda}\right) \frac{2K_{\text{loss}}(w_0)}{g^2(w_0)} \int_{-\infty}^{t} \int_\infty j_0(w, t') f_\perp(t, t') g(\vec{T}(w)) dw dt' \\ &= -\left(\frac{c}{\omega_\lambda}\right) \frac{2K_{\text{loss}}(w_0)}{g^2(w_0)} \int_{-\infty}^{t} \int_\infty j_1^{(0)}(\vec{z}, t') f_\perp(t, t') g(\vec{T}(\mathbf{T}^{-1}\vec{z})) d\vec{z} dt' \end{aligned} \tag{44}$$

$$\vec{T}(w) = \mathbf{T}(\vec{z} + \vec{n}(w)) \tag{45}$$

$$g(\vec{T}(\mathbf{T}^{-1}\vec{z})) = g(\vec{z} + \mathbf{T}\vec{n}(\mathbf{T}^{-1}\vec{z})) = g(\vec{z}) + (\nabla_\perp g)^T \mathbf{T}\vec{n}(\mathbf{T}^{-1}\vec{z}) \tag{46}$$

$$\vec{n}(\vec{z}) = \frac{e}{c} \nabla_\perp g(w) V_\perp(t) \tag{47}$$

$$V_\perp(t) = -\left(\frac{e}{\omega_\lambda}\right) \frac{2K_{\text{loss}}(w_0)}{g^2(w_0)} \int_{-\infty}^{t} \int_\infty j_1^{(0)}(\vec{z}, t') f_\perp(t, t') (\nabla_\perp g)^T \mathbf{T} \nabla_\perp g(\mathbf{T}^{-1}\vec{z}) V_\perp(t - t_r) d\vec{z} dt' \tag{48}$$

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