Particle Accelerators
Vol. 54, pp 193-202, 1996

RIGOROUS LOWER BOUNDS ON THE SURVIVAL TIME IN PARTICLE ACCELERATORS

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Analyzing stability of particle motion in storage rings is an interesting question in the general field of stability analysis in weakly nonlinear motion. A method which we call pseudo invariant estimation (PIE) is used to compute lower bounds on the survival time in circular accelerators. The pseudo invariants needed for this approach are computed via nonlinear perturbative normal form theory. Differential Algebraic (DA) techniques are essential to manipulate the Taylor expansions required in this theory. Using the new method of Differential Algebra with Remainder (RDA), the remainder terms in Taylor expansions can be bounded rigorously during numerical calculations, which will ultimately lead to a rigorous bound on the survival time. The lower bounds on the survival times are large enough to be relevant; the same is true for the lower bound on the dynamic aperture of a storage ring, which can also be computed.

KEY WORDS: long-term stability, nonlinear dynamics, Nekhoroshev, Ljapunov, RDA, interval arithmetic, global optimization

1 INTRODUCTION

Estimating the time of stable motion for planetary systems has first started the interest in the stability of weakly nonlinear mechanical systems about 100 years ago; in our modern days this question became important in accelerator physics with the analysis of the dynamics in storage rings. One of the prominent current examples, the Large Hadron Collider (LHC) at CERN, will have to allow particles to circle the 27 km long tunnel for one day at one millionth of a percent less than the speed of light in order to make effective high energy physics experiments; this corresponds to 10^9 orbits around the ring.

In the past, the question of long term stability in storage rings has been analyzed by various methods including kick tracking[1], element by element tracking and one–turn map tracking[2, 3], symplectic long term generating function tracking[4, 5], approximately symplectic tracking[6], evaluation of Lyapunov exponents and tune shift analysis[7], as well as Nekhoroshev estimates[8].

* Supported in part by the National Science Foundation and the Alfred P. Sloan Foundation

The principle underlying the proof of the Nekhoroshev estimate[9] as well as similar approaches based on conventional Ljapunov stability theory is to numerically obtain estimates of the quality of pseudo invariance, which then translate into bounds for survival time [10]. We call this the pseudo invariant estimation (PIE) method[11, 12]. Although some of these methods are useful analysis tools, they all fail to give mathematically rigorous lower bounds on the time particles stay inside the storage ring, because of the need to determine bounds of highly complex functions and the difficulty in accounting for all approximation errors. We here describe an application of the PIE Method[13] which, when combined with the so-called RDA methods currently under development[14, 15] will ultimately yield strict and rigorous bounds on the survival time.

The one turn transfer map $\vec{M}$ of a storage ring maps initial phase space coordinates $\vec{z}_i$ of a particle into final coordinates $\vec{z}_f$ after one turn around the ring. This transfer map can also depend on a set of parameters $\vec{\delta}$ of the accelerator, $\vec{z}_f = \vec{M}(\vec{z}_i, \vec{\delta})$. The Taylor expansion to some order n of the transfer map in respect to $\vec{z}$ and $\vec{\delta}$ can be computed by the so called DA method[16, 17].

By a normal form transformation[18, 19] $\vec{A}$, a map $\vec{M}$ is transformed into a normal form map $\vec{N} = \vec{A} \circ \vec{M} \circ \vec{A}^{-1}$. The normal form transformation $\vec{A}$ is chosen to let the Taylor expansion of $\vec{N}$ have as many vanishing Taylor coefficients as possible. For symplectic maps, the normal form transformation also yields functions $f(\vec{z})$ which are invariants of the map $\vec{M}$ up to order $n+1$. This means that the Taylor expansions of $f \circ \vec{M}(\vec{z})$ and $f(\vec{z})$ agree up to order $n+1$. For weakly nonlinear dynamics, where the Taylor expansion represents the map $\vec{M}$ quite well, these functions are approximate invariants, or so-called pseudo invariants. This is the case for particles moving close to the central periodic orbit in a storage ring.

A pseudo invariant $f(\vec{z})$ can be used as a measure for the distance from a particle with coordinates $\vec{z}$ to the central periodic orbit. In linear accelerator theory, such an invariant could be the sum of the different emittances. The pseudo invariant can be used to describe a region in phase space $\mathcal{A} = \{\vec{z} | f(\vec{z}) \leq \epsilon\}$. In an application of the map $\vec{M}$ the so described distance from the origin changes by the deviation function $d(\vec{z}) = f(\vec{M}(\vec{z})) - f(\vec{z})$.

If an upper bound $\bar{\delta}$ on $d(\vec{z})$ in the phase space volume $\mathcal{A}$ can be found, it can rigorously be said how far a particle in $\mathcal{A}$ can move away from the closed orbit in one turn. It can also be stated rigorously that particles in the phase space volume $\mathcal{O} = \{\vec{z} | f(\vec{z}) \leq \epsilon - N\delta\}$ will not leave the volume $\mathcal{A}$ for N applications of the map. Or if $\mathcal{O} = \{\vec{z} | f(\vec{z}) \leq \epsilon_0\}$, one can state that no particle in $\mathcal{O}$ has left the region $\mathcal{A}$ after $N = (\epsilon - \epsilon_0)/\delta$ applications of the Map. One can now try to improve the dynamical properties of a storage ring by maximizing the number N of turns thus guaranteed to be stable.

2 RIGOROUS PSEUDO INVARIANT ESTIMATION

The PIE method relies on a suitable pseudo invariant $f(\vec{z})$ which describes a particle's distance from the central orbit, on the deviation function $d(\vec{z}) = f(\vec{M}) - f$, and on an upper bound on $d(\vec{z})$. The main challenge dealt with in this paper is to find an upper bound $\bar{\delta}$ on the increase $d(\vec{z})$ of $f(\vec{z})$ during a particle's motion once around the ring.

Conventional interval arithmetic is very useful for rigorous and verified global optimization[20, 21, 22, 23]. If a function d is defined on an interval I and can be evaluated on a computer, then interval arithmetic can be used to find an interval $D(I) \supset \{d(x)|x \in I\}$ which contains all the values of d on I. The upper bound of $D(I)$ is therefore a strict bound on the global maximum of d. Unfortunately, often $D(I)$ overestimates $\{d(x)|x \in I\}$ substantially, a phenomenon known as interval blow–up. In the early stages of this work, several methods of standard interval optimization have been studied. However, it turned out that the function $d = f \circ \vec{M} - f$ for which we have to find an upper bound is far more complex than those in typical applications of interval optimization. The deviation function $d(\vec{z})$ is a multivariate polynomial of order $n \times (n+1)$, when n is the evaluation order of the normal form transformation. This complexity leads to estimated computation times of several million years if conventional interval arithmetic is applied and blow–up is suppressed sufficiently by subdividing the phase space region $\mathcal{A}$ into many intervals. Nevertheless, the use of interval arithmetic seems imperative for any rigorous treatment of the stability problem, since any tracking method only tests a small part of phase space of measure zero.

In the weakly nonlinear systems of interest, the blow–up in the conventional methods is mainly due to the fact that computing $d(\vec{z})$ involves subtraction of the two big numbers $f \circ \vec{M}(\vec{z})$ and $f(\vec{z})$, an operation which is especially prone to interval blow–up. There is a very large number of operations required for the evaluation of d, which provides additional but very much smaller blow up. The method of Differential Algebra with Remainder allows to avoid almost all of the blow up, since the bulk of the functional dependencies are carried in the form of Taylor coefficients, and blow up is limited to the Taylor remainder terms, which are many orders of magnitude smaller. Finally a bound for the Taylor polynomial itself has to be found. However, for the special case of normal form invariants, a significant simplification arises from the fact that the Taylor polynomial of the invariant defect d vanishes completely up to order n. In this case, another method that is more straightforward to implement yet far less flexible can be used, namely the method of Interval Chains[13].

3 PARAMETRIZING THE REGIONS OF PHASE SPACE

To avoid interval blow–up, it is advisable to perform as few operations as possible. We therefore try to minimize the computations required to represent the initial region $\mathcal{O}$ and the allowed region $\mathcal{A}$. The conventional invariants η_i of linear motion in D degrees of freedom define the invariant tori of first order. We now write the tori in linear normal form space where the section of the tori in the $z_{2i-1} \times z_{2i}$ coordinate planes are circles

$$\mathcal{T}_{NF}(\vec{\eta}) = \{\vec{z}| \begin{pmatrix} z_{2i-1} \\ z_{2i} \end{pmatrix} = \sqrt{\eta_i} \begin{pmatrix} \cos(\phi_i) \\ \sin(\phi_i) \end{pmatrix}, \phi_i \in [0, 2\pi], \forall i \in \{1, \dots, D\}\} \ . \quad (1)$$

The allowed region in normal form space with acceptances ϵ_i for the i^{th} degree of freedom is then written as

$$\mathcal{A}_{NF} = \{\vec{z}|\vec{z} \in \mathcal{T}_{NF}(\vec{\eta}), \sum_{i=1}^{d} \frac{\eta_i}{\epsilon_i} \leq 1\} \ . \quad (2)$$

With the linear normal form transformation $\vec{A}_1$, we get the allowed region

$$\mathcal{A} = \{\vec{A}_1^{-1}(\vec{z})|\vec{z} \in \mathcal{A}_{NF}\} \ . \quad (3)$$

To bound the deviation function δ on the surface of the acceptance, we have to find

$$\begin{aligned} \delta &= \max\{f \circ \vec{M} - f|\mathcal{A}\} \\ &= \max\{(f \circ \vec{M} - f) \circ \vec{A}_1^{-1}|\mathcal{A}_{NF}\} \\ &= \max\{f \circ \vec{A}_1^{-1} \circ \vec{A}_1 \circ \vec{M} \circ \vec{A}_1^{-1} - f \circ \vec{A}_1^{-1}|\mathcal{A}_{NF}\} \ . \quad (4) \end{aligned}$$

The map $\vec{A}_1 \circ \vec{M} \circ \vec{A}_1^{-1}$ is the transfer map in the first order normal form space and is to first order a rotation in every $z_{2i-1} \times z_{2i}$ plane in phase space. Call this linear rotation map $\vec{R}$ and let $\vec{N} = \vec{A}_1 \circ \vec{M} \circ \vec{A}_1^{-1} \circ \vec{R}^{-1}$; $\vec{N}$ is the identity to first order. Write $g = f \circ \vec{A}_1^{-1}$ to get

$$\delta = \max\{g \circ \vec{N} \circ \vec{R} - g|\mathcal{A}_{NF}\} \ . \quad (5)$$

The polynomial g of order $n+1$ does not contribute to the Taylor remainder, which is bound by RDA; and therefore the second appearance of g in equation (5) can be omitted. The rotation $\vec{R}$ leaves the tori invariant and can therefore be avoided; we finally get the simple RDA evaluation of

$$\delta = \max\{g \circ \vec{N}|\mathcal{A}_{NF}\} \ . \quad (6)$$

Now the intervals $[0, 2\pi]$ in the definition of $\mathcal{T}_{NF}$ in equation (1) have to be covered by many small intervals. The maximum upper bound of all intervals is a rigorous upper bound for $d(\vec{z})$. All blow–up due to linear transformations and to low order compensations is avoided in this way.

4 COMPARISON BETWEEN INTERVALS AND RDA

Several nonlinear systems were studied using the RDA method to provide upper bounds for the invariant defects. In order to get a sense for the quality of these upper bounds, the numbers are compared with approximations for the maximal invariant defects obtained by a rather tight rastering of $d(\vec{z})$ in real arithmetic. Because of the large number of local maxima, this method proved to be the most robust non–interval approach to estimate the absolute maxima of the functions involved. Lower bounds on the number of stable turns obtained by conventional intervals are given in the tables (1) and (2) in order to illustrate the usefulness of RDA. When conventional intervals were used, the deviation function was simplified as much as possible by accounting for cancelations up to second order analytically. The number of conventional intervals and the number of RDA vectors used in the bounding are equivalent.

TABLE 1: Predictions of the number of stable turns for the Henon map at tune 0.13, strength parameter 1.1, and starting position $(x, a) = (0.01, 0)$ as a function of the order of the normal form transformation.

Order of Invariant	Interval Bounding (guaranteed)	DA with Remainder (guaranteed)	Conventional Rastering (optimistic)
2	895	891	1,086
3	1,736	9,926	11,450
4	1,668	54,016	65,667
5	1,674	678,725	809,612
6	1,670	3,389,641	4,351,679
7	1,671	42,640,927	52,474,387
8	1,671	192,650,961	263,904,035

To provide a first illustration of the method, we chose the Henon map, which is often used as crude model of a storage ring. The results of these calculations are shown in table (1). The number of predicted turns increases with order, since the quality of the pseudo invariance increases. In the case of interval bounding, the number of guaranteed turns saturates around 1700 and does not increase further with order since low order cancelation dominates the blow–up. The superiority of rigorous bounding with RDA is obvious.

For table (2) a realistic accelerator, the Los Alamos PSR II, was analyzed. The same data are shown as for the previous, more academic example; however, a limitation arises from the fact that the remainder terms in the transfer map are ignored. In section 6 it will be described how this limitation can be overcome with RDA. To limit the calculation time for our example, the intervals were 5 times as wide as the intervals used for the previous table.

TABLE 2: Predictions of the number of stable turns as a function of the order of the approximate invariant for the Los Alamos PSR II storage ring. The emittances were 100 mm mrad.

Order of Invariant	Interval Bounding (guaranteed)	DA with Remainder (guaranteed)	Conventional Rastering (optimistic)
3	179	16,137	38,385
4	179	18,197	38,857
5	173	309,356	560,309
6	173	347,312	613,135
7	171	925,531	2,184,998
8	171	1,004,387	2,248,621

5 CONSIDERING UNKNOWN PARAMETERS OF THE SYSTEM

So far, the normal form method assumes that the one–turn map of the storage ring in question is well known. Since this is rarely the case, the theory has been extended to maps which depend on an unknown parameter. Neither particle energy nor the magnet parameters are known exactly and have to be treated as parameters which can not be accurately specified. If the map $\vec{M}$ is a function of a parameter, then the nonlinear normal form transformation $\vec{A}$ also depends on a parameter. With DA programs one can compute the Taylor expansion of this parameter dependent map $\vec{A}$. The pseudo invariant $f(\vec{z})$ then also depends on the parameter. Changing the parameter changes the transfer map and the pseudo invariant simultaneously, such that $f(\vec{z})$ stays a good pseudo invariant for a wide range of the parameter.

Without parameters, the volume was covered by 8,000,000 intervals; for the parameter dependent case, 50,000,000 intervals were used. The results are shown in table (3). Instead of the field strength, also other parameters could have been used, like the uncertainty of the particle's energy or uncertainty in the length of an element.

TABLE 3: Lower bounds on the turns of particles for an initial emittance of one half the acceptance, obtained by the RDA–PIE method.

Lower bound for	**IUCF Ring**	**PSR II**
Simplest application	19,500,358	58,680,622
Field uncertainty 1% / 0.01%	10,768,020	45,819,009

6 OUTLOOK: USING TAYLOR MAPS WITH REMAINDER BOUND

So far we were only concerned with nonlinear motion which is described by Taylor maps. The number of interest was the survival time of particles in an accelerator. This time was formulated as the number of map applications for which no phase space point of the initial beam distribution is mapped into a forbidden region. A method was presented with which rigorous lower bounds on this number can be obtained. In the phase space regions which we analyzed, the Taylor maps usually describe the accelerator well and the limits obtained are valuable for storage rings, however, strictly speaking they are not rigorous, since the transfer map of an accelerator is not a polynomial map, but itself has a remainder.

While it can be estimated that this missing remainder term does not have a significant effect, a fully rigorous treatment of the stability question requires its complete consideration. This is only possible using a full implementation of RDA as well as efficient methods to bound the remainders in the integration process, which are currently under development. In order to provide estimates on what remainders to expect, we compare an 8^{th} order map with the 12^{th} order map, and find an interval bounding the difference. The results for the examples are shown in table (4).

TABLE 4: Maximum estimated error of the Taylor map in the phase space region of interest.

IUCF Ring	PSR II
$[-0.248, 0.248] \cdot 10^{-13}$	$[-0.282, 0.282] \cdot 10^{-11}$

The influence of this remainder term on the actual stability estimate is minor; table (5) shows the results of the same calculations shown in table (3), but with inclusion of the estimated remainder bound.

TABLE 5: Results of the RDA–PIE bounds on the survival time of particle motion for rigorous description of the systems by a Taylor map with estimated remainder bound.

	Guaranteed Turns	With Unknown Parameter
IUCF	19,441,816	10,749,711
PSR II	49,241,881	39,571,014

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