REFINEMENT OF THE NORMAL FORM METHOD FOR LONG TERM STABILITY ESTIMATES

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Abstract

First, the normal form method of obtaining long term stability estimates for particle motion in storage rings is recapitulated with an emphasis on utilizing nonlinear normal form invariants. Two methods will be introduced which make the obtainable bounds on long term stability more optimistic, one by separating phase space in appropriate regions, the other by involving multi–turn maps. So far, the normal form method assumes that the one turn map of the storage ring in question is well known. Since this is rarely the case, the theory has been extended to maps which depend on an unknown parameter. The applicability will be demonstrated by the transfer map of the Indiana University storage ring for two degrees of freedom.

Introduction

The normal form method was occasionally called Nekhoroshev method, which bares its name from work only indirectly related to the method which will be presented [1,2,3]. The line of thought that will be described was first used by Warnock et al. [4,5,6], who observed that in one part of Nekhoroshev's proof of exponential estimates a similar thought is used. The underlying idea is very simple. Assume two disconnected regions in phase space, $\mathcal{N}$ and $\mathcal{P}$. If a smooth function f on phase space has an upper bound $\overline{f}$ in $\mathcal{N}$ which is smaller than a lower bound $\underline{f}$ in $\mathcal{P}$ then on every curve connecting $\mathcal{N}$ and $\mathcal{P}$ the function f has to change by at least $\Delta f = \overline{f} - \underline{f}$. If the deviation function of a phase space map $\vec{M}$ is defined in respect to f as $\delta = f \circ M - f$ and has an upper bound $\overline{\delta}$ on phase space then one needs at least $\Delta f / \overline{\delta}$ applications of the map $\vec{M}$ to map elements of $\mathcal{N}$ into $\mathcal{P}$.

For the precise domains $\mathcal{N}$ and $\mathcal{P}$ of interest in accelerator physics and for the optimizations required for $\underline{f}$, $\overline{f}$, and $\overline{\delta}$, please refer to [7]. $\mathcal{N}$ is associated with the initial region of the beam and $\mathcal{P}$ with the forbidden region, the region where particles can not survive one turn in the storage ring. In order to estimate long survival times, $\overline{\delta}$ should be small, and therefore f should be an approximate or pseudo invariant of motion.

Nonlinear normal form theory is applied to obtain pseudo invariants. Normal form theory attempts to simplify a map on phase space by transforming coordinates. The coordinate basis in which the map has the simplest form is called normal form basis. In [8] it was shown how the transformation to this basis can be found for Taylor maps. It turns out that the simplest form can be achieved if the linear tune is not on a resonance of order lower than the order of the Taylor expansions involved. If for $2n$ dimensional phase space $\vec{A}$ is a symplectic transformation into normal form basis of a symplectic map $\vec{M}$ then $(A_i^2 + A_{i+n}^2)$, $i \in \{1, \dots, n\}$ are n invariants of motion, one for each degree of freedom. The motion in normal form space therefore lies on invariant tori. Due to the Taylor approach, the quality of this invariant decreases with distance form the origin. This is especially true if the motion is not integrable and the linear tunes are close to a resonance, since then the high order Taylor coefficients in $\vec{A}$ increase with proximity to the resonance.

Parametrizing the initial and the allowed region

We can choose the function f as a suitable function of the pseudo invariants. In [7] the motivation of choosing the following f is described.

$$f = \sum_{i=1}^{n} \frac{A_i^2 + A_{i+n}^2}{\epsilon_i} \tag{1}$$

where ϵ_i is a weight factor. To represent the initial beam well by $\mathcal{N}$. We choose ϵ_i to be the beam emittances in the different degrees of freedom and the region $\mathcal{N}$ and $\mathcal{P}$ by

$$\mathcal{N} = \{\vec{z} | f(\vec{z}) \le 1\} , \quad \mathcal{P} = \{\vec{z} | f(\vec{z}) > \alpha\} , \quad \alpha > 1 \quad . \tag{2}$$
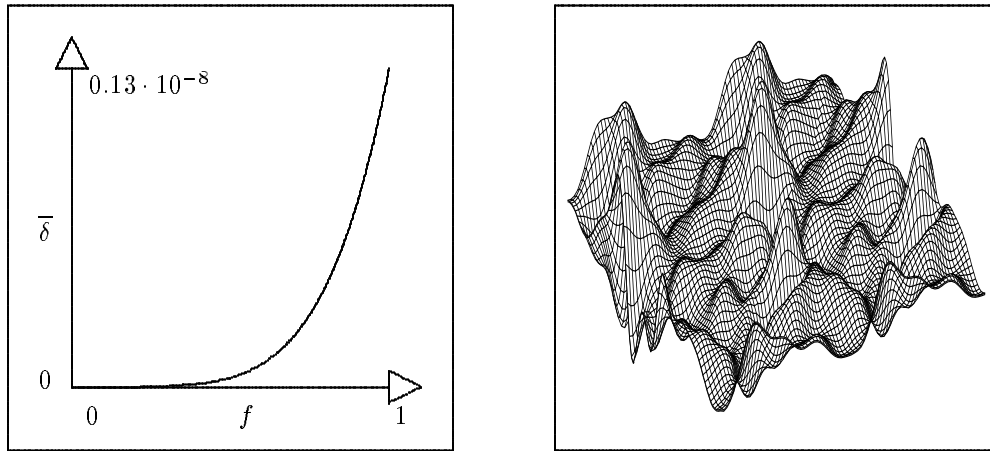

Figure 1: a: δ increases with f, b: δ is a quite smooth function on phase space. Maximizing by scanning is therefore reliable.

With the final and initial phase space known, δ has to be found in the region in which particles move between leaving $\mathcal{N}$ and reaching $\mathcal{P}$. As shown in figure 1a, δ usually increases with f and therefore it is quite accurate to assume that the maximum of δ occurs at the surface with $f = \alpha$. The optimization is performed by scanning this surface. Figure 1b shows the function δ on a two dimensional phase space. The smoothness of this function gives reason to belive that scanning is an appropriate method of finding an approximate global maximum.

The first line in table 1 gives a lower bond for the turn numbers of particles starting in an emittances of 2πmm mrad. The number of turns particles should survive in order to accumulate enough current is 3700000000. This number of turns can, however, not be guaranteed for the whole desired phase space region. The second columns of table 1 gives the emittances for which survival can be guaranteed by this method. In the following, two methods are described which improve the guaranteed number of turns.

Dividing phase space

Because of the rapid increase of δ with increasing f, it is appropriate to separate the regions between $f = 1$ and $f = \alpha$ by surfaces with $f = \alpha_i$, $i \in \{1, \ldots, n\}$. Then the lower bound on the number of turns will

become

$$N = \sum_{i=1}^{n} \frac{\alpha_i - \alpha_{i-1}}{\bar{\delta}_i} \; .$$ (3)

The turn numbers obtained from this technique and the transportable emittances is given in table 1 in the second line.
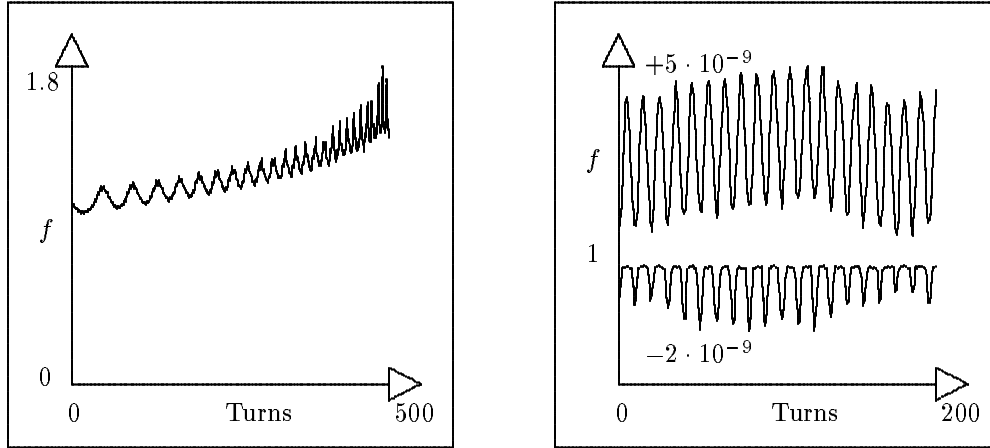
Time evaluation

Figure 2: a: slow increase of the pseudo invariant of motion over many turns, b: periodic change of the pseudo invariant of motion after relatively few turns.

Figure 2a shows how close the pseudo invariants are to invariants of motion. Over many turns the quantity of f changes but it always oscillates around a mean which grows very slowly. The normal form method, however, bounds the growth of the pseudo invariant by considering the biggest growth that can happen in one application of the map. As demonstrated in figure 2b, the biggest growth that can be generated by N applications of the map is usually much bigger than N times the biggest growth that can happen during a single turn. Therefore it is advantageous to consider the maximum growth that can occur when the map is applied M times where M corresponds to the number of map applications that leads from one waist in figure 2b to another.

In the third line of table 1 shows the guaranteed stable turns for the desired emittances and the guaranteed stable emittances of the Indiana ring for certain optimized settings of magnet parameters and for a certain particle energy.

Parameter dependence

Since neither particle energy nor the magnet parameters are known exactly, we analyzed the dependence of stability on certain parameters. We computed the map as a function of a parameter of interest and then scanned not only through the interesting region of phase space, but also through the interval in which the parameter could lie in reality.

Table 1 shows in its fourth line the number of stable turns and the stable emittances for an energy variation of $\pm 0.3\%$, which is a typical energy spread of the beam in the Indiana ring.

The field setting of all dipoles was varied by random factors of relative strength 0.1% . The last line of the table contains the number of turns particles will at least move around the ring for any setting of the magnets in this range.

Method	turns for needed emittances	emittances for needed turns
Simplest application	15191870	0.51 πmm mrad
Divided phase space	103644000	0.82 πmm mrad
Multi–turn maps	3076246000	1.91 πmm mrad
Energy as parameter	16606040	0.08 πmm mrad
Fields as parameter	1148088000	1.42 πmm mrad

Table 1: Lower bounds on the turns of particles for initial emittance of 2π mm mrad and lower bound on the stable emittances for 3700000000 turns obtained by various variations of the normal form method. To limit the computation time, we evaluated 7^n points for n relevant phase space dimensions.

We belive that the described method gives a very reliable lower bound on the numbers of stable turns. To make completely rigorous statements about guaranteed bounds, several steps have to be performed in a more rigorous way. Especially, it will not be sufficient to approximate the maximum of δ by scanning. Interval arithmetic methods which guarantee a global maximum have to be used [9,10]. Utilizing this guaranteed optimization is the subject of another paper [11].

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