

Investigation of the flat-beam model of the beam-beam interaction

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At the interaction point of a storage ring collider each beam is subject to perturbations due to the electromagnetic field of the counterrotating beam. For flat beams, a well-known approximation models the beam by a current sheet which is uniform in the horizontal plane, restricting the particle motion to the vertical direction. In this classical model a water-bag beam distribution has been used to find working points and beam-beam tune shift parameters which lead to a stable beam distribution. We investigate the stability of a more realistic Gaussian equilibrium distribution. A linearized Vlasov equation written in action-angle variables is used to compute the radial and angular modes of a perturbation in two-dimensional phase space to first order in the displacement from the design trajectory. We find that the radial modes, which are often neglected, can have a stabilizing effect on the beam motion.

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I INTRODUCTION

Colliding particle bunches in a storage ring exert an electromagnetic force on each other. The beam-beam parameter ξ is the tune shift exerted by one bunch on a particle near the center of the opposing bunch. It is a useful measure of the strength of the beam-beam interaction. A limiting value of ξ_y is reached in an e^+e^- collider when further increases in beam intensity lead to particle loss or to an increase in the vertical emittance of the beam. In e^+e^- colliders, where the action of radiation excitation and damping produce a flat beam, the observed vertical beam-beam parameter limit is in the approximate range $0.02 \leq \xi_y \leq 0.1$ [1,2]. At present it is not known whether the emittance increase is due to an incoherent, single-particle effect or to a coherent, collective instability of the colliding beams. The DCI storage rings at LAL, Orsay, France, used a pair of e^+ and e^- beams to collide with another pair, in an attempt to cancel the beam-beam force [3]. It was found, however, that the beam-beam limit in DCI was not significantly improved by the charge cancellation. Derbenev [4] explained this result in terms of a collective instability of the four-beam system and in [5] the performance of DCI was analyzed numerically. This suggests that the beam-beam limit for two-beam e^+e^- colliders may also be due to a collective instability. Simulations in [6–8] show collective oscillations of the beam at the beam-beam limit.

In Refs. [4,9,10] the stability of the colliding beams was examined by solving the Vlasov equation for an equilibrium distribution with small perturbations. Chao and Ruth [10] considered a beam-beam model in which motion was confined to the vertical plane, and in which the beam has a “water-bag” equilibrium distribution

(uniform within an ellipse in phase space). Synchrotron radiation damping and excitation were not considered. When the Vlasov equation was solved for a linearized beam-beam force, coherent beam modes were found to be unstable near each resonance. In [9] the stability of a Gaussian equilibrium distribution was analyzed with the Vlasov equation for round beams where the beam-beam force can be expanded in Bessel functions. A flat beam model with a Gaussian distribution and synchrotron radiation was studied in [11,12] under the assumption that the distribution always remains Gaussian. A similar approach was chosen in [13] for a purely linear beam-beam force. The findings of these models, e.g., flip-flop solutions and period- n solutions, are verified numerically in [14] where the behavior of flat and round beams is considered as well.

In this paper we extend the model of Chao and Ruth to a Gaussian equilibrium distribution. In Sec. II we set up the equations of motion for the phase space distribution and its perturbations and linearize the beam-beam force. In Sec. III we solve the equations of motion for radial and angular modes up to first order in the displacement from the design trajectory and discuss the implications of our results.

II. BEAM EVOLUTION

We model the flat beam as a current sheet which is uniform in the horizontal direction, x , and consider only motion in the vertical direction, y . Consider one-dimensional phase space distributions $\psi_1(y, y', s)$ and $\psi_2(y, y', s)$ of the two beams which are normalized to unity. Then the deflection from the second (first) beam on a particle in the first (second) beam is

$$\Delta y'_{1,2} = -I_{\psi_{2,1}}(y, s), \quad (1)$$

where we define

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$$I_{\psi}(y, s) \equiv \frac{4\pi N r_e}{\gamma} \int_{-\infty}^{\infty} d\bar{y} \operatorname{sgn}(y - \bar{y}) \int_{-\infty}^{\infty} d\bar{y}' \psi(\bar{y}, \bar{y}', s), \quad (2)$$

and N is the number of particles per unit width in x and r_e is the classical radius of the electron. Both beams are assumed to have the same number of particles per unit width. The equations describing the motion of $\psi_{1,2}$ are given by the two Vlasov equations

$$\frac{\partial \psi_{1,2}}{\partial s} + y' \frac{\partial \psi_{1,2}}{\partial y} - K(s)y \frac{\partial \psi_{1,2}}{\partial y'} - \frac{\partial \psi_{1,2}}{\partial y'} \delta_p(s) I_{\psi_{2,1}}(y, s) = 0, \quad (3)$$

where the periodic delta function and the unperturbed focusing function are denoted by $\delta_p(s)$ and $K(s)$, respectively. We want to determine whether the beam is stable. That is, we want to know if small perturbations of the phase space density grow. Thus, we choose a perturbative ansatz

$$\psi_{1,2} = \psi_0 + \Delta\psi_{1,2}, \quad (4)$$

where ψ_0 is the equilibrium distribution, i.e., a solution of Eq. (3) with $\psi_1(y, y', s) = \psi_2(y, y', s) = \psi_0(y, y', s) = \psi_0(y, y', s + C)$, where the circumference of the ring is denoted by C . Substituting Eq. (4) into Eq. (3), subtracting Eq. (3) written for the equilibrium distribution, and neglecting the term which contains a product of two perturbations we find

$$\frac{\partial \Delta\psi_{1,2}}{\partial s} + y' \frac{\partial \Delta\psi_{1,2}}{\partial y} - \frac{\partial \Delta\psi_{1,2}}{\partial y'} F(y, s) - \delta_p(s) \frac{\partial \psi_0}{\partial y'} I_{\Delta\psi_{2,1}} = 0, \quad (5)$$

where

$$F(y, s) = K(s)y + \delta_p(s) I_{\psi_0}(y). \quad (6)$$

If we approximate the beam-beam force as linear in y

$$F(y, s) \approx F(s)y = K(s)y + \delta_p(s) I_{\psi_0}^1 \quad (7)$$

with

$$I_{\psi_0}^1 = I_{\psi_0}(0) + \left. \frac{\partial}{\partial y} I_{\psi_0}(y) \right|_{y=0} \cdot y. \quad (8)$$

we can replace $K(s)$ by the perturbed focusing function $F(s)$ to compute the perturbed Twiss parameters. In the next step we transform Eq. (5) to action-angle coordinates

$$y = \sqrt{2\beta J} \cos\phi, \quad y' = -\sqrt{2\beta J} \frac{\sin\phi + \alpha \cos\phi}{\beta}. \quad (9)$$

The betatron function is perturbed by the linearized beam-beam kick from ψ_0 . We form the linear combinations for the σ and the π modes

$$f_{\pm} = \Delta\psi_1 \pm \Delta\psi_2. \quad (10)$$

Then Eq. (5) can be decoupled and rewritten in action-angle coordinates as

$$\frac{\partial f_{\pm}}{\partial s} + \frac{1}{\beta} \frac{\partial f_{\pm}}{\partial \phi} \mp \delta_p(s) \frac{\partial \psi_0}{\partial y'} I_{f_{\pm}}^1 = 0. \quad (11)$$

The quantity $(\partial \psi_0 / \partial y') = -\sqrt{2\beta J} [\sin\phi (\partial / \partial J) \psi_0 + (\cos\phi / 2J) (\partial / \partial \phi) \psi_0]$ simplifies since the linearization of the beam-beam force in Eq. (7) leads to $\psi_0 = \psi_0(J)$ and we are left with

$$\frac{\partial f_{\pm}}{\partial s} + \frac{1}{\beta} \frac{\partial f_{\pm}}{\partial \phi} \pm \sqrt{2\beta J} \sin\phi \delta_p(s) \frac{\partial \psi_0}{\partial J} I_{f_{\pm}}^1 = 0. \quad (12)$$

In the following discussion we omit the label \pm .

III. SOLVING THE EQUATIONS OF MOTION

When the interaction term in Eq. (3) is not considered, any differentiable distribution which depends solely on J is an equilibrium distribution. In general, ψ_0 will be a function of both J and ϕ . Fortunately, an arbitrary differentiable function of J is an equilibrium distribution, at least to linear order in y after introducing the perturbed betatron function. We choose a Gaussian equilibrium distribution

$$\psi_0(J) = \frac{1}{2\pi\epsilon} e^{-(J/\epsilon)}, \quad (13)$$

since in the presence of damping and quantum excitation the beam distribution naturally tends to a Gaussian distribution. The deflection of a particle due to the presence of a Gaussian beam can be obtained from Eq. (2),

$$I_{\psi_0}(y) = \frac{4\pi N r_e}{\gamma} \operatorname{erf}\left(\frac{y}{\sqrt{2\beta\epsilon}}\right). \quad (14)$$

We expand the linearized version of Eq. (12) using the ansatz

$$f(J, \phi, s) = \sum_{n'=0}^{\infty} \sum_{l'=-\infty}^{\infty} g_{n'l'}(s) e^{-(J/\epsilon)} L_{n'}\left(\frac{J}{\epsilon}\right) e^{il'\phi}. \quad (15)$$

Since the perturbation must be periodic in ϕ we can express the ϕ dependence in terms of a Fourier series. The orthogonality relation for the Laguerre polynomials comes with the convenient weight factor $e^{-(J/\epsilon)}$ which simplifies working with expressions that contain the Gaussian equilibrium distribution. Furthermore, using the weight factor in the set of basis functions guarantees that the perturbation falls off as $J \rightarrow \infty$. We will refer to the modes represented by the first and second index in g_{nl} as "radial" modes and "angular" modes, respectively, i.e., these words refer to the two-dimensional phase space described by action-angle variables. With Eq. (15) the linearization in Eq. (8) leads to

$$I_f \propto \int_{-\infty}^{\infty} d\bar{y}' \left[\int_{-\infty}^y d\bar{y} f(\bar{y}, \bar{y}', s) - \int_y^{\infty} d\bar{y} f(\bar{y}, \bar{y}', s) \right], \quad (16)$$

$$I_f^1 \propto \int_{-\infty}^{\infty} d\bar{y}' \left[\int_{-\infty}^0 d\bar{y} f(\bar{y}, \bar{y}', s) - \int_0^{\infty} d\bar{y} f(\bar{y}, \bar{y}', s) + 2yf(0, \bar{y}', s) \right]. \quad (17)$$

Using $\int_0^{\infty} e^{-x} L_n(x) dx = \delta_{n0}$ the first part of I_f^1 is given by

$$\int_0^{\infty} dJ \int_{-\pi/2}^{\pi/2} d\phi [f(J, \phi + \pi, s) - f(J, \phi, s)] = -4\epsilon \sum_{l'=-\infty}^{\infty} g_{0(2l'+1)} \frac{(-1)^{l'}}{2l'+1}. \quad (18)$$

The second part is given by

$$2 \int_{-\infty}^{\infty} d\bar{y}' f(0, \bar{y}', s) = 2 \int_0^{\infty} dJ \frac{1}{\sqrt{2\beta J}} [f(J, \pi/2, s) + f(J, -\pi/2, s)] = \sqrt{4\pi} \sqrt{\frac{2\epsilon}{\beta}} \sum_{n'=0}^{\infty} \sum_{l'=-\infty}^{\infty} g_{n'2l'} (-1)^{l'} \frac{(2n)!}{(2^{n'} n'!)^2}. \quad (19)$$

Here we have made use of

$$\int_0^{\infty} \frac{1}{\sqrt{x}} e^{-x} L_n(x) dx = \frac{(2n)!}{(2^n n!)^2} \equiv \sqrt{\pi} P_n. \quad (20)$$

Inserting I_f^1 into Eq. (12), projecting this equation onto our chosen set of basis functions by means of the orthogonality relation of the Laguerre polynomials

$$\int_0^{\infty} e^{-x} L_n(x) L_m(x) dx = \delta_{nm} \quad (21)$$

and using

$$\begin{aligned} \int_0^{\infty} \sqrt{x} e^{-x} L_n(x) dx &= -\frac{(2n)! \sqrt{\pi}}{2(2n-1)(2^n n!)^2} \\ &= -\frac{\sqrt{\pi}}{2(2n-1)} P_n \end{aligned} \quad (22)$$

and

$$\int_0^{\infty} x e^{-x} L_n(x) dx = \delta_{n0} - \delta_{n1}, \quad (23)$$

we obtain

$$\frac{\partial g_{nl}}{\partial s} + \frac{il}{\beta} g_{nl} = \mp \delta_p(s) \xi \sum_{n'=0}^{\infty} \sum_{l'=-\infty}^{\infty} M_{nl, n'l'} g_{n'l'}, \quad (24)$$

where

$$\begin{aligned} M_{nl, n'l'} &\equiv i2\pi \left[\frac{1}{2n-1} P_n(\delta_{l,1} - \delta_{l,-1}) \delta_{n',0} (-1)^{(l'-1/2)} \frac{1}{l'} a_{l'} \right. \\ &\quad \left. + (\delta_{n,0} - \delta_{n,1}) (\delta_{l,2} - \delta_{l,-2}) P_{n'}(-1)^{(l'/2)} b_{l'} \right]. \end{aligned} \quad (25)$$

and

$$\xi = \frac{Nr_e}{\gamma} \sqrt{\frac{2\beta^*}{\pi\epsilon}}.$$

The coefficients a_l are 1 for odd l and 0 for even l and vice versa for the coefficients b_l . Each column and each row of the matrix M refers to one particular combination of an n and an l value.

IV. DYNAMIC TUNE

We calculate the tune ν in terms of the unperturbed tune ν_0 by means of Eq. (26).

$$\nu - \nu_0 = \frac{1}{4\pi} \oint \beta(s) (F(s) - K(s)) ds. \quad (26)$$

In order to obtain $F(s) - K(s)$ the deflection in Eq. (14) is linearized. This gives

$$\nu - \nu_0 = \frac{Nr_e}{\gamma} \sqrt{\frac{2\beta^*}{\pi\epsilon}} \equiv \xi, \quad (27)$$

where β^* denotes the beta function at the interaction point.

V. COHERENT BEAM-BEAM INSTABILITY

We solve the ordinary differential equation (24) and rewrite the solution in matrix form such that the beam transport after one turn is described by a matrix T which acts on a column vector G that contains all g_{nl} , i.e., $G(C) = TG(0)$. We parametrize the beam current by the linear tune shift parameter ξ . One obtains the following relation for the g_{nl} 's immediately before and immediately after the interaction point by integrating through the interaction point:

$$G(0^+) - G(0^-) = \pm \xi M G(0^-). \quad (28)$$

There is no coupling among different Fourier components between collisions. In this case Eq. (24) simplifies to

$$\frac{\partial g_{nl}}{\partial s} + \frac{il}{\beta(s)} g_{nl} = 0, \quad (29)$$

which is solved by

$$g_{nl}(C^-) = g_{nl}(0^+) e^{-il \int_0^C [1/\beta(s)] ds} = g_{nl}(0^+) e^{-2\pi i l \nu}. \quad (30)$$

The one-turn transfer matrix becomes

$$T_{\pm} = R(\mathbb{1} \pm \xi M), \quad (31)$$

where R is a diagonal matrix which has the elements $e^{-2\pi i l \nu}$ on its diagonal. The matrix M has the following

properties which follow immediately from Eq. (25):

$$\begin{aligned}
 M_{nl,n'l'} &= 0 \quad \text{for } l + l' = \text{odd}, & M_{nl,n'-l'} &= M_{nl,n'l'}, \\
 M_{n-l,n'l'} &= -M_{nl,n'l'}, & M_{nl,n'l'}^* &= -M_{nl,n'l'}.
 \end{aligned}
 \tag{32}$$

In order to decide whether the system is stable or not we have to find out what happens to an arbitrary initial perturbation after a large number of turns, i.e., one needs to consider the limit T^N where $N \rightarrow \infty$. Every matrix norm of the latter quantity tends to infinity if the absolute value of one eigenvalue of T is bigger than 1. To analyze the stability for a given tune ν and a beam-beam parameter ξ , we therefore compute the eigenvalue λ_{\max} that has the largest modulus. In case of instability we compute the corresponding eigenvector G and find its component g_{nl} which has the largest modulus. This indicates that the instability mainly drives the radial mode n and angular mode l , causing f to be dominated by $L_n(J/\epsilon)e^{il\phi}$. Since the perturbation f must be real taking its complex conjugate must leave f invariant which gives the constraint $g_{nl} = g_{n-l}^*$. Indeed Eq. (24) is invariant under complex conjugation and replacing $l \rightarrow -l$. It follows that the coefficients of T have the property $T_{n-l,n'-l'} = T_{nl,n'l'}$, which also follows from Eq. (32). This requires that eigenvalues of T are either real or come in a pair with their complex conjugate: Let S be a matrix performing the transformation $l \rightarrow -l$ then we have $STSSG = \lambda SG$ and finally $T(SG^*) = \lambda^*(SG^*)$. Therefore, the l mode and the $-l$ mode are always excited simultaneously with equal strength.

VI. RESULTS AND DISCUSSION

In Figs. 1 and 2 we varied the tune ν between 0 and 1 and the beam-beam parameter ξ between 0 and 0.12. A point has been plotted if the absolute value of all eigenvalues of T is smaller than or equal to 1 for both the σ and the π modes. We truncated T to the indicated modes. In Fig. 1 only the five modes $l = -2, \dots, 2$ for $n = 0$ were considered. In Fig. 2 we included the same angular modes for $n = 0, \dots, 2$. The first and second order resonances can be recognized clearly. Resonances of orders higher than 2 cannot be expected in our linearized model. It is interesting to note that the inclusion of radial modes stabilizes the motion of the beam so that a larger ξ can be tolerated.

In Figs. 3 and 4 we again varied ν and ξ and plotted the largest eigenvalue $|\lambda_{\max}|$ vs ν and determined which mode becomes unstable by selecting the biggest component of the eigenvector which is associated with the largest eigenvalue. The plot shows that in the absence of dynamics in the radial direction $l = \pm 1$ and $l = \pm 2$ modes become unstable in the vicinity of $\nu = 0.5$, but in Fig. 4 only $l = \pm 1$ modes are excited around $\nu = 0.5$. Furthermore, the unstable $l = \pm 2$ modes which accumulate in the vicinity of $\nu = 0.25$ and $\nu = 0.75$ are attenuated if the $n = 1$ mode is included. Therefore, the radial motion leads to a damping of the $l = \pm 2$ modes.

In Fig. 5 we computed the phase of the largest eigenvalue of $l = \pm 2$ instabilities, corresponding to quadrupole oscillations (π mode only), versus the perturbed tune for various $\Delta\nu$. The slope of the two lower lines is 2 which indicates that the collective oscillation frequency of the

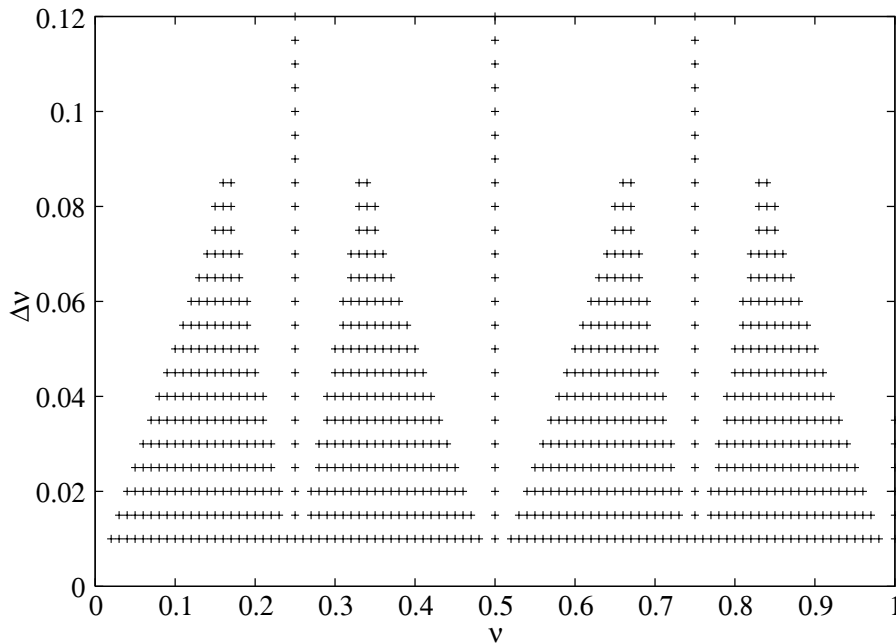
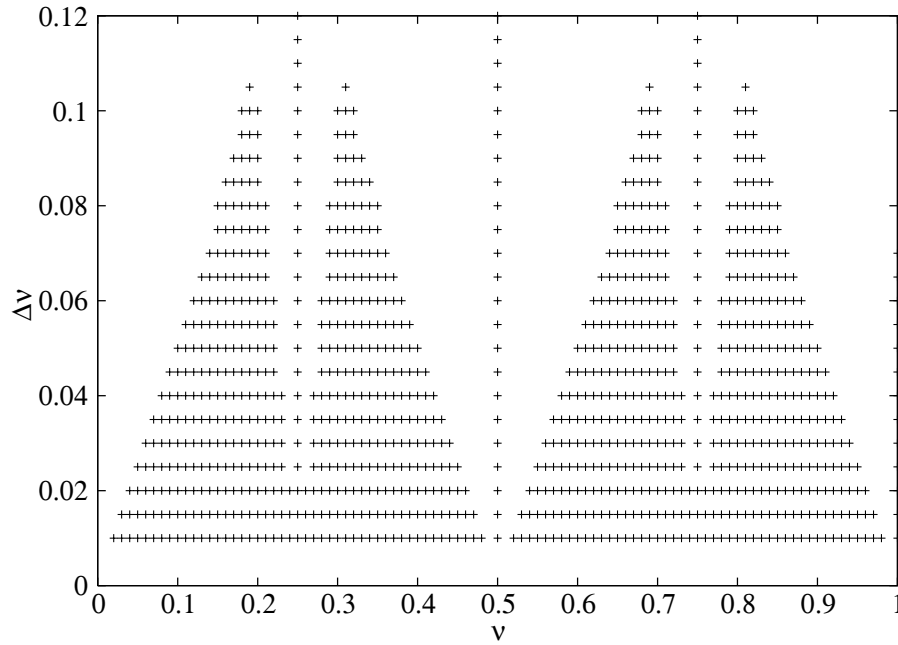


FIG. 1. Stability diagram for $n = 0, l = -2, \dots, 2$.


 FIG. 2. Stability diagram for $n = 0, \dots, 2, l = -2, \dots, 2$.

quadrupole mode is twice the single-particle oscillation frequency for small ξ . The spread of the points for fixed ν shows how strongly the beam-beam parameter ξ influences the frequency of quadrupole oscillations. In Fig. 6 this spread is significantly lower which again shows that radial modes have a stabilizing effect.

The dependence of this spread on ν can be understood analytically. For simplicity we consider only the $n = 0$

modes. Close to a resonance where $l\nu$ is integer, g_{0l} and g_{0-l} perturb the beams the most. Thus, we content ourselves with the following 2×2 matrix [10]:

$$T = \begin{pmatrix} e^{-2\pi i l \Delta} & 0 \\ 0 & e^{2\pi i l \Delta} \end{pmatrix} \left[\mathbb{1} \pm i\alpha \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right], \quad (33)$$

which satisfies all properties listed in Eq. (32) for $i\alpha =$

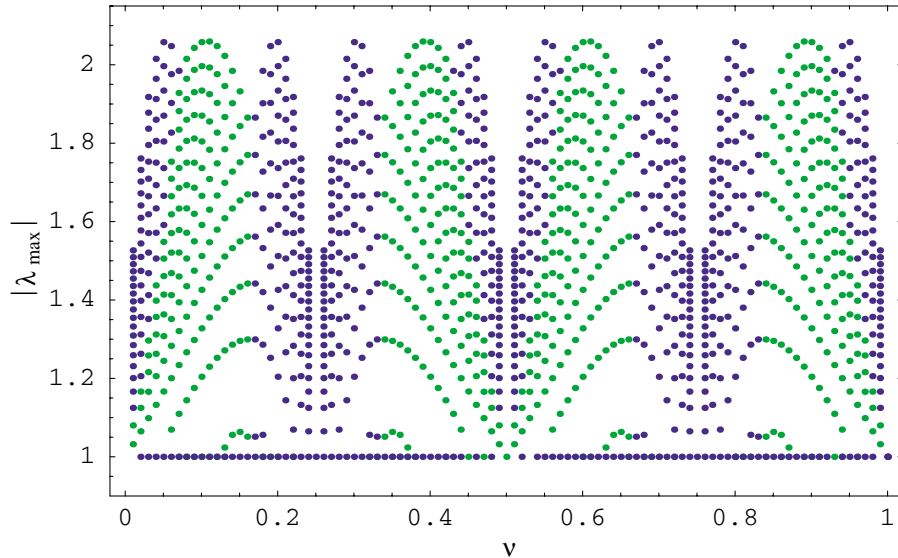


FIG. 3. (Color) Absolute value of the largest eigenvalue λ_{\max} vs tune. Light (green) points indicate unstable $l = \pm 1$ modes and dark (blue) points indicate unstable $l = \pm 2$ modes. The following modes were included: $n = 0, l = -2, \dots, 2$.

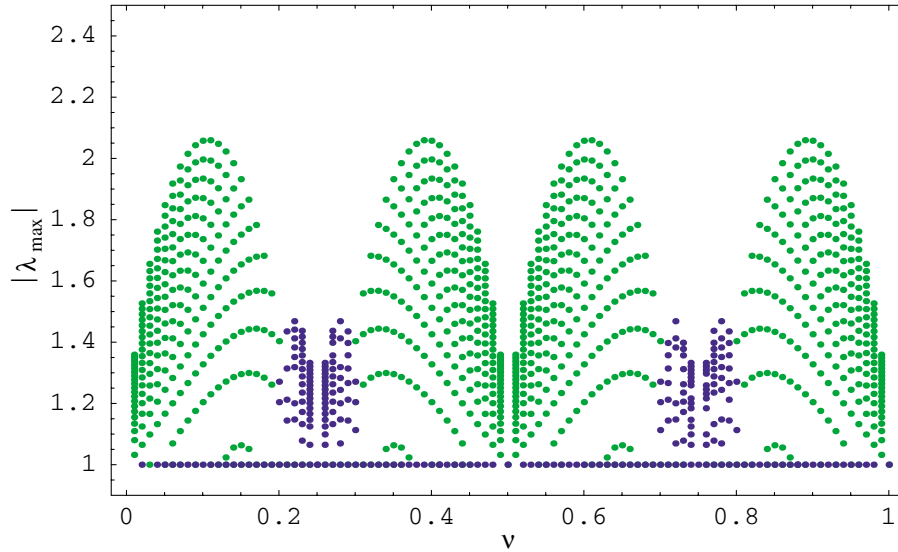


FIG. 4. (Color) Same as Fig. 3, but for $n = 0, \dots, 1, l = -2, \dots, 2$.

$\xi M_{0l,0l}$. The imaginary parts of the eigenvalues of the matrix T vanish for eigenvalues whose absolute value is bigger than 1. This leads to the plateaus at 0 and 0.5 in Figs. 5 and 6 at tunes ν where the $l = \pm 2$ mode becomes unstable in Figs. 3 and 4.

The difference between the dipole oscillation frequencies ν_π plotted in Fig. 7 light (green) and ν_σ plotted dark (blue) of the π and the σ mode divided by the beam-beam parameter ξ is referred to as the Meller factor [15] or the Yokoya factor [16]. This factor is plotted for all points of our computation for which both the π and the σ modes indicate stable motion. In Fig. 8, one can see that this factor is always above 1.25 in our Gaussian flat beam model.

There are only a few points close to $\nu = 0.25$ and $\nu = 0.75$ since the $l = 2$ modes for these tunes are unstable for small ξ .

VII. POSSIBLE EXTENSIONS

A. Higher order resonances

In order to study resonances of order higher than 2 Eq. (2) must not be linearized, but rather the double integral has to be expanded about $y = 0$ to orders higher than 1. The expansion to second order contains $y^2 \int_{-\infty}^{\infty} d\bar{y}' (d/dy) f(\bar{y}, \bar{y}', s)|_{\bar{y}=0}$. Inserting the expansion in Eq. (15) for f and writing (d/dy) in terms of J and ϕ allows the evaluation of the integral. The resulting term

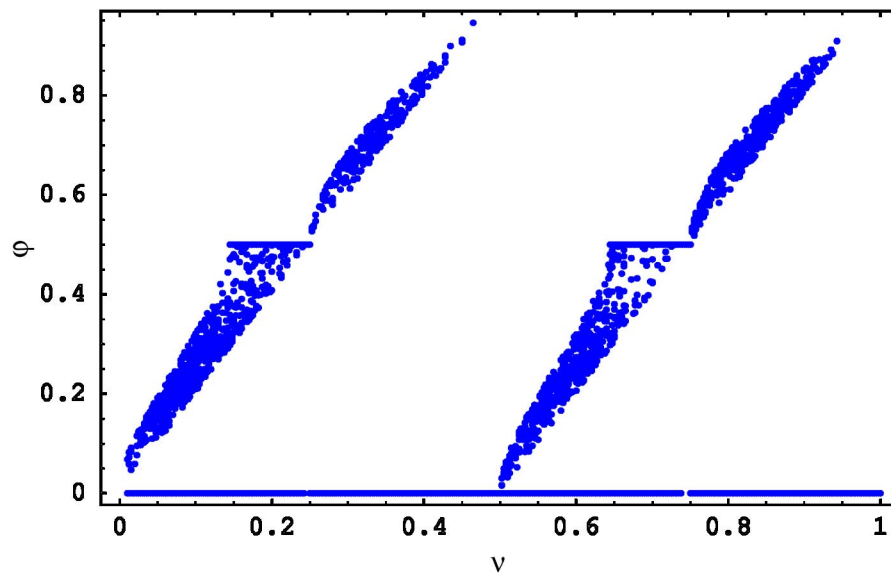


FIG. 5. (Color) Phase versus perturbed tune for $n = 0, l = \pm 2$ modes (π mode only).

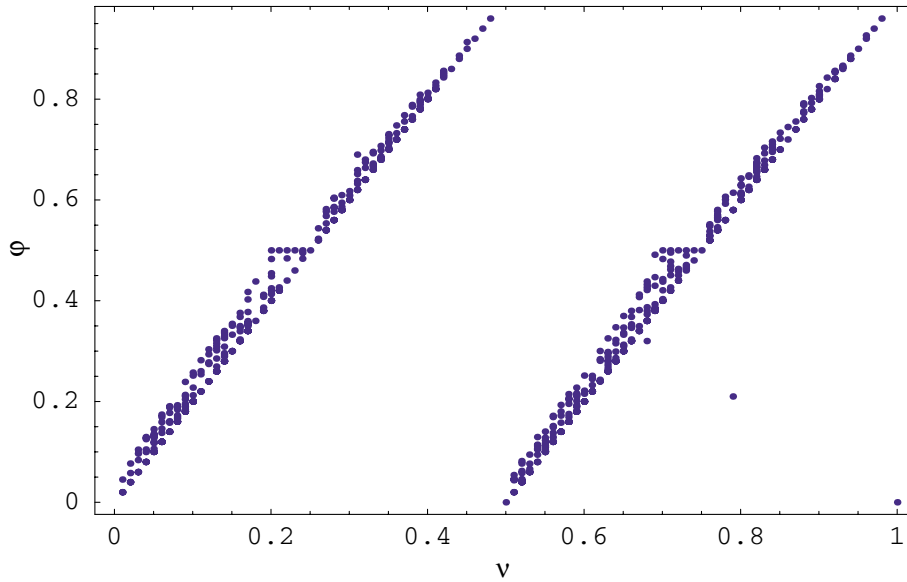


FIG. 6. (Color) Phase versus perturbed tune for $n = 0, \dots, 1$, $l = \pm 2$ modes (π mode only).

$(\partial\psi_0/\partial y')y^2 = -(1/2\pi\epsilon^2)(2\beta J)^{3/2}e^{-J/\epsilon} \sin\phi\cos^2\phi$ in Eq. (5) needs to be expanded in Laguerre polynomials and gives rise to higher orders in radial modes. The n th order term can be written in terms of powers of \sqrt{J} , $\cos n\phi$, $\sin n\phi$, and lower frequency parts. Since the beam-beam force acts only at a single point, its contribution is not averaged out in the limit of a large number of turns if the tune matches the frequency of one of the sine or cosine functions. This is the case if the tune is a rational number, so higher order resonances would appear in Fig. 2. Without truncating the series the model would result in an infinite number of resonances

since one can always find a rational number between two irrational numbers. However, this procedure is complicated by the fact that Eq. (13) is not an equilibrium distribution anymore when nonlinear terms are included.

When the length of the bunch and its longitudinal motion are included, synchrotron resonances can occur [17] when the bunch length is in the order of the betatron function. Including these resonances would require an extension of our treatment from two- to four-dimensional phase space. This would be a worthwhile but tedious continuation of our work.

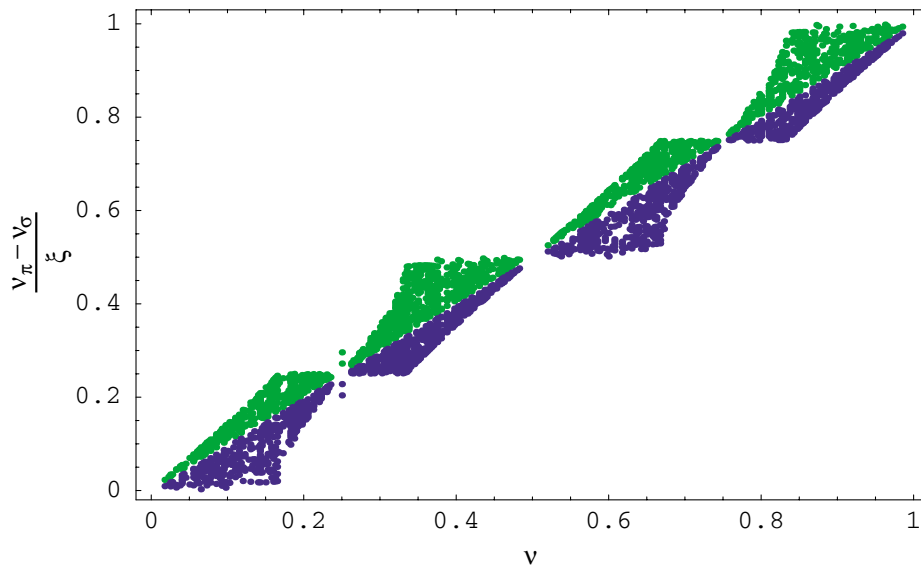


FIG. 7. (Color) The dipole oscillation frequencies are plotted light (green) for the f_+ distribution and dark (blue) for the f_- distribution with $\xi = 0$ to 0.2 for $n = 0$, $l = \pm 1$ modes.

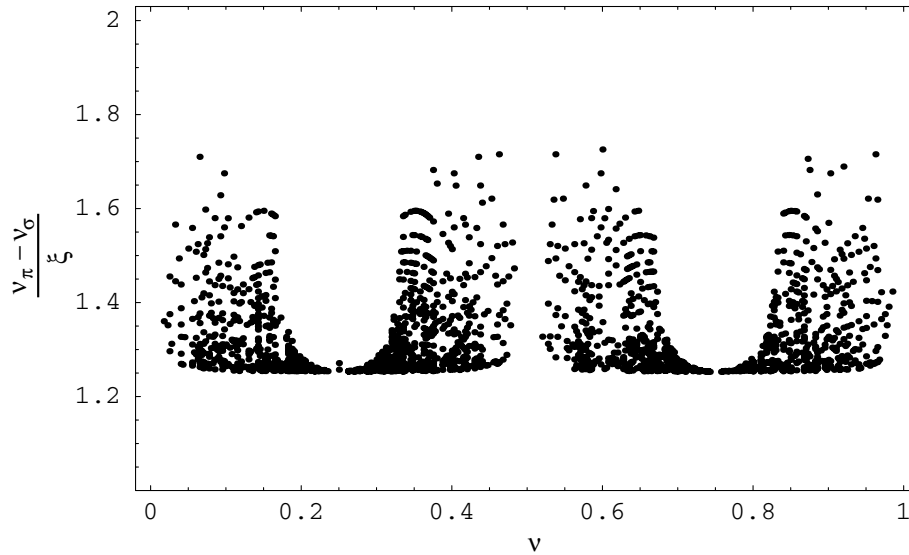


FIG. 8. The Meller factor for stable motion in the region $\xi = 0$ to 0.2 for $n = 0$, $l = \pm 1$ modes.

B. Damping by synchrotron radiation

One can extend the presented model to account for damping by synchrotron radiation. In order to obtain the equilibrium distribution in Eq. (13) quantum excitation must be included as well. This turns Eq. (3) into the Fokker-Planck equation (34). In preliminary computations we found that the graphs we presented above remain unchanged for realistic values of the damping

and excitation coefficients. To simplify the Fokker-Planck equation, we averaged over the phases in the damping and excitation terms but not in the beam-beam interaction term. This can be justified since the betatron phases in the terms for damping and quantum excitation change during one turn while the phase in the interaction term changes only once per turn. In Eq. (34) λ is the energy loss per turn due to synchrotron radiation divided by the energy of the particle, η is the dispersion, and D is the quantum excitation coefficient.

$$\begin{aligned} \frac{\partial \psi_{1,2}}{\partial s} + y' \frac{\partial \psi_{1,2}}{\partial y} - \left(\frac{\lambda}{C} y' + K(s)y + \frac{4\pi N r_e}{\gamma} \delta_p(s) I_{\psi_{2,1}}(y, s) \right) \frac{\partial \psi_{1,2}}{\partial y'} \\ = \frac{\lambda}{C} \psi_{1,2} + D \left(\eta \frac{\partial}{\partial y} + \eta' \frac{\partial}{\partial y'} \right)^2 \psi_{1,2}. \end{aligned} \quad (34)$$

C. Different tunes

If the two beams have different tunes, Eq. (10) cannot be used anymore to decouple the system. It is easier to work with the uncoupled system and solve for the g_{nl} of the two beams separately. Introducing the column vector G which contains the g_{nl} for both beams, one can proceed as before and describe the beam transport for each turn by a matrix multiplication with a matrix T . Introducing

$$\tilde{R} = \begin{pmatrix} R(\nu_1) & 0 \\ 0 & R(\nu_2) \end{pmatrix}, \quad (35)$$

where $R(\nu)$ is a diagonal matrix which has the components $e^{-2\pi i l \nu}$ we can write the matrix T as

$$T = \tilde{R} \left[\mathbb{1} + \xi \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} \right]. \quad (36)$$

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