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3.8 Beam Dynamics Activities at UNM

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Here we report on beam dynamics research in the Mathematics and Statistics Department of the University of New Mexico.

There are many significant problems in beam dynamics that are at the forefront of what is understood in dynamical systems, stochastic processes and statistics, and in scientific and high performance computing. This makes it an ideal area for interaction between universities and accelerator laboratories.

The following are people with whom we are currently working: Alejandro Aceves (AA) (Mathematics at UNM), Desmond Barber (DB) (DESY), Pat Colestock (PC) (LANL), Scott Dumas (SD) (Mathematics at U. of Cincinnati), Klaus Heineman (KH) (DESY), Georg Hoffstaetter (GH) (DESY), Bill Sáenz (BS) (Catholic U., NRL), Tanaji Sen (TS) (FNAL), Linda Klamp Spentzouris (LS) (FNAL, IIT), and Bob Warnock (BW) (SLAC).

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Our activities have several foci: Vlasov and Vlasov–Fokker–Planck equations and properties of their solutions for the beam–beam interaction, and for longitudinal collective effects due to wake fields; spin dynamics; stochastic processes and perturbation techniques.

3.8.1 Vlasov and Vlasov–Fokker–Planck Equation

3.8.1.1 (1) Numerical

BW and JE have developed the Perron–Frobenius (PF) method for integrating the Vlasov–Fokker–Planck equation [1, 2] and have applied it to two problems. MV, TS and JE are developing the

Weighted Macro-Particle Tracking (WMPT) method in the context of the strong-strong beam-beam [3]. We are extending these algorithms from one to two degrees of freedom (d.o.f.) and pursuing parallel implementation.

3.8.1.2 (2) Analytical

This includes the existence of equilibria (periodic solutions), linearized behavior about these solutions (a generalized Floquet theory) and the associated linear stability and Landau damping, development of a weakly nonlinear theory and three wave coupling, and a study of fully nonlinear effects such as solitary waves, nonlinear Landau damping and turbulence. The weakly nonlinear theory requires a complete solution of the linearized equation. The work of Landau and Case-Van Kampen solves this for the basic plasma problem and we hope to extend these approaches using the the work of Bart and Warnock on integral equations of the third kind [4]. Some progress on the existence of equilibria and solitary waves will be discussed below. In certain situations the Vlasov equation can be approximated by “hydrodynamic” or moment equations and analyzed as above.

3.8.2 Longitudinal Collective Effects Due to Wakefields

3.8.2.1 (1) Bunched beam with radiation

BW and JE made a major advance in simulating the saw-tooth instability in the SLAC damping rings with a realistic wakefield [1]. The Vlasov-Fokker-Planck (VFP) theory was used and good agreement was found with several aspects of observations, including the presence of the bursting (sawtooth) mode with period comparable to the damping time. A special method was needed to treat the Vlasov part of the operator, as the straightforward finite difference approach was unstable. We were successful with the PF approach which is based on the method of characteristics and which allowed us to go to realistic damping times with smooth (low noise) densities. Almost the same method was applied to the beam-beam as discussed in the next section. Analytically, we are trying to understand the details of our numerical results and studying the linearization about the Haïssinski solution using the third kind integral equation. We are extending this work to include a multiturn cavity wake and a nonlinear applied R.F.

3.8.2.2 (2) Coasting beam without radiation

We are extending the pioneering work of Colestock, Spentzouris and Tzenov for proton beams [5]. The work on bunched beams should be directly applicable and we will proceed as discussed in the section on the Vlasov and VFP equation. On the one hand this problem is simpler than the bunched beam case but on the other hand we may encounter numerical stability problems because of no radiation. This is joint work with AA, PC, LS, MV, BW, IV and JE and will form the basis of a Ph.D. thesis by IV. The hydrodynamic approximation has been investigated for soliton behavior by both Tzenov and AA, and AA has derived a KdV equation by multiple time scales.

3.8.2.3 (3) Calculation of longitudinal wake field in a special case

BW and NF have determined analytically the longitudinal wake fields in a tube with smoothly varying radius and finitely conductive wall [6].

3.8.2.4 (4) *A simple plasma problem*

MV and JE have designed a small project as a senior thesis for NF. Her project is to summarize what is known related to the equations of ion acoustic waves in plasmas [7]. She is proceeding along the lines outlined in the Vlasov and VFP section and is using ideas from Sattinger's article including his discussion of the pseudospectral codes for integrating the KdV equation. What we learn may be useful for the more complicated beam dynamics problems. Bisognano discusses a similar problem in the context of beam dynamics.

3.8.3 Beam–Beam Interaction

3.8.3.1 (1) *Beam–beam with radiation using PF*

The equilibrium (periodic) phase space distribution of stored colliding electron beams has been studied from the viewpoint of VFP theory using the PF method [2]. Numerical integration of the VFP system in one degree of freedom revealed a nearly Gaussian equilibrium with non–diagonal covariance matrix. This result is reproduced approximately in an analytic theory based on linearization of the beam–beam force (the VFP equation remains nonlinear). Analysis of an integral equation for the equilibrium distribution, without linearization, establishes the existence of a unique equilibrium at sufficiently small current. The role of damping and quantum noise is clarified through a new representation of the propagator of the linear Fokker–Planck equation with harmonic force. Details are discussed in [2]. The proof of the existence of the equilibrium uses an implicit function theorem in an appropriate Banach space and will be published separately by BW and JE.

With MV, we are investigating the behavior of the code without the radiation and with smaller beam–beam parameter for use in the proton case. The code is being extended to 2–d.o.f. and to parallel algorithms. The existence of an equilibrium for the linear beam–beam force has been demonstrated [2] and can also be viewed in terms of a self–consistently determined dynamic beta (as pointed out by MV and others). Our implicit function theorem works for arbitrarily small radiation but in the limit of no radiation there are surely resonance issues and the existence of an equilibrium is an open question. Perturbation theory looks promising for proving the existence of a quasi–equilibrium.

3.8.3.2 (2) *Beam–beam for hadrons using WMPT*

MV, TS and JE have developed a second code [3] for the simulation of beam–beam effects in the strong–strong regime for protons (i.e. without radiation). The code is based on symplectic tracking of macro–particles which are assigned a weight representing the phase space density at their initial conditions. We refer to this as weighted macro–particle tracking (WMPT). We found that this idea, while developed independently, has been used by Wollman [8] on the basic plasma problem. We have treated three different 1–d.o.f. limits of the beam–beam force. Particular emphasis was put on development of efficient algorithms for the computation of the beam–beam kick. With this simulation code we have so far studied the time evolution of the moments of the density and in particular the frequency spectrum of the centroid motion. We are investigating the possible coupling of the π and σ modes and also the existence of analogous modes when the tunes are separated. The code will be extended to 2–d.o.f..

3.8.3.3 (3) *Weak–Strong Beam–Beam with fluctuations*

We are investigating the impact of fluctuating fields on emittance growth in the presence of the weak–strong beam–beam in a hadron collider. We model fluctuations by an Ornstein–Uhlenbeck process and then analytically derive diffusion coefficients in action in the presence of three different sources of fluctuations: fluctuations in offsets between the beams, tune fluctuations and fluctuations in beam size. Integration of the diffusion Fokker–Planck equation with these diffusion coefficients leads to predictions on emittance growth. We apply our results to the LHC. This is work by TS, with help from Maria–Paz Zorzano, IV and JE and is an extension of the 1–d.o.f. case [9].

3.8.4 Spin Dynamics

3.8.4.1 (1) *Spin tune*

The paper “A quasiperiodic treatment of spin motion in storage rings — a new perspective on spin tune” with DB, KH and JE is nearly complete. In this paper we show how spin motion on the periodic closed orbit of a storage ring can be analyzed in terms of the Floquet theorem for equations of motion with periodic parameters. The spin tune on the closed orbit emerges as an extra frequency of the system which is contained in the Floquet exponent in analogy with the wave vector in the Bloch wave functions for electrons in periodic atomic structures. We then proceed to show how to analyze spin motion on quasi–periodic synchro–betatron orbits in terms of a generalization of the Floquet theorem and we find that, provided small divisors are controlled by applying a Diophantine condition, a spin tune can again be defined and that it again emerges as an extra frequency in a Floquet–like exponent. We thereby obtain a deeper insight into the concept of “spin tune” and the conditions for its existence.

3.8.4.2 (2) *Existence of a stationary polarization*

KH and JE are applying ergodic theorems to stroboscopic averaging of Liouville densities and spin fields. A class of orbital systems with volume preserving flows is defined. Performing stroboscopic averaging and applying the Birkhoff ergodic theorem, Liouville densities, periodic in the azimuthal variable, are obtained. More importantly, particles with both spin and integrable orbital motion are considered. By performing stroboscopic averaging on the polarization densities one gets, via the Birkhoff and Von Neumann Ergodic Theorems, polarization densities which are periodic in the azimuthal variable. This demonstrates that the tracking algorithm, encoded in the program SPRINT and used in the simulation of spin polarized storage rings, is mathematically well founded.

3.8.4.3 (3) *Adiabatic invariance of the invariant spin field*

GH has written a *Habilitation* thesis [10] on modern concepts of spin dynamics in accelerators. SD and JE are collaborating with GH on a theoretical aspect of this work; techniques from rigorous averaging theory are being used to directly show that the invariant spin field is an adiabatic invariant under slow changes in parameters such as beam energy. This requires special care because many resonances are crossed during the acceleration process. We have an elementary proof in a special case related to the closed orbit and are optimistic with regard to its extension to phase space trajectories.

3.8.4.4 (4) Polarized protons for the VLHC

MV has analyzed spin motion in the combined function lattice of the 2×20 TeV so called stage 1 low field collider. The results suggest the possibility that polarized protons at about 80 times the RHIC energy can be achieved with reasonable effort.

3.8.5 Stochastic Averaging and Single Realization Noise

In [11], JE introduced the single realization problem and argued in a relevant example that single realization noise plus nonlinearity can lead to diffusion. This was done by developing a new stochastic averaging formalism and applying it in a heuristic way. However, we now have a proof in the very special case of [12, Example 8.1, pages 257–259] which gives us added confidence in the approach.

3.8.6 Perturbation theory

It is the view of JE that the KAM and Nekhoroshev theorems are great foundational works in dynamical systems but they are mostly irrelevant to beam dynamics because the perturbation parameter, call it ϵ , must be much, much smaller than typical beam dynamics parameters. The KAM theorem is valid for infinite times on Cantor-like sets whereas Nekhoroshev theorems are valid for exponentially long times on open sets or even on all of phase space. For rigorous results, we believe first and second order averaging theorems (where there are only modest restrictions on ϵ and which give error bounds) are much more applicable. First order averaging theorems give $O(\epsilon)$ error bounds at $O(1/\epsilon)$ times whereas second order theorems can give $O(\epsilon^2)$ errors at $O(1/\epsilon)$ times or in special cases $O(\epsilon)$ errors at $O(1/\epsilon^2)$ times. Jeng Shih and JE discussed such theorems with beam dynamics in mind in [13] and we are pursuing several extensions and complements.

Theorem 2 on p. 595 of [13] requires the convergence of two series which suffers from the small divisor problem (i.e. the series will only converge on a Cantor set in ω space). We have modified this condition using the work of SD on “cut-off Diophantine conditions” (See [14]) which gives a much more satisfying condition on an open set of ω values.

The above theorems are basically about trajectories. However in beam dynamics we are primarily interested in quasi-invariants. BS, SD and JE are extending the averaging results to the case where invariants of an associated unperturbed problem are quasi-invariants for the full problem, at $O(1/\epsilon^2)$ times.

In many problems the perturbation is a delta function and standard averaging theorems do not apply. We have developed a set of averaging theorems for maps, to include this kick case.

Because the averaging theorems require ϵ to be small, it is important to know how small in a given context. BW and JE plan to investigate this numerically using the recent work [15].

We are extending our averaging approaches to the Vlasov equation both in terms of the collective force and in terms of the weakly nonlinear theory.

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