Nonlinear spin transfer maps

C. Weissbaecker[a] and G. H. Hoffstaetter[b]

[a] TU-Darmstadt, Institut für Angewandte Physik, Hochschulstr. 6, D-64289 Darmstadt, FRG
[b] Deutsches Elektronen-Synchrotron DESY, Notkestrasse 85, D-22603 Hamburg, FRG

Abstract: An analytical solution of the spin orbit equation of motion up to a
given order in the initial phase space coordinates is derived for an arbitrary electro-
magnetic field. An iterative procedure successively leads from a first order expan-
sion to the required order in the initial coordinates. Finally a simple example is
presented.

1 Introduction

While a particle moves through the electro-magnetic fields of a particle optical device, its phase
space coordinates change. This particle transport is described by the transfer map $\vec{M}$ which
maps initial phase space coordinates $\vec{z}_i$ into final coordinates $\vec{z}_f = \vec{M}(\vec{z}_i)$ after the device. In
the field of accelerator physics, the Transport formalism [1] is well established. It approximates
the transfer map $\vec{M}$ by its Taylor expansion to a given order, typically two or three, in $\vec{z}_i$.
The Transport formalism possesses three limitations: the particle moves in a plane, the coor-
dinates are not canonical and the fields are given in the SCOFF (Sharp-Cut-Off-Fringe-Field)
approximation if no additional fringe field approximation is used. Subsequently a formalism
which overcomes these limitations [2] is applied to iteratively determine nonlinear components
of phase space motion. The motion of the classical spin of a particle in an accelerator depends
on the particles trajectory and thus on $\vec{z}_i$. So far only the linear parts of the spin motion in
every accelerator element are used to compute and discuss the spin motion in HERA [3, 4].
Nonlinear effects are only found by the combination of several elements. In a flat ring the spin
of a proton with 920 GeV precesses 1756 times about the vertical direction during one turn.
This rapid rotation illustrates the requirement of an exact computation of the spin motion.
Therefore we are interested in the nonlinear contribution of a single particle optical element.

The following well-known equations of motion have to be solved:

1. The Lorentz force equation which describes the motion of the particle with charge q in
   phase space,

$$\frac{d\vec{p}}{dt} = -\frac{q}{m\gamma}\left(\vec{B} \times \vec{p} - m\gamma\vec{E}\right) \ .$$

(1)

2. The Thomas Bargmann-Michel-Telegdi (T-BMT) equation which describes the spin motion of a particle traveling along a certain phase space curve [5],

$$\frac{d\vec{s}}{dt} = -\frac{q}{m\gamma}\left[(1+a\gamma)\vec{B}_\perp + (1+a)\vec{B}_\parallel - \left(a+\frac{1}{\gamma+1}\right)\frac{\vec{p}}{mc^2}\times\vec{E}\right]\times\vec{s}. \tag{2}$$

In a transverse magnetic field the spin equation of motion differs from the phase space motion by a factor $(1+a\gamma)$ which leads to the above mentioned $a\gamma = 1756$ spin rotations during one turn at the HERA energy of 920 GeV. We first solve the orbital equation of motion and will then insert the results in the T-BMT equation to compute the spin motion.

We choose a right handed orthogonal coordinate system with which particle motion close to some reference curve $R(l)$ can be expressed. The reference curve is parameterized by its path length l. This curvilinear coordinate system based on the unit vectors $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_l$ in the x, y, and l direction differs from the Frenet-Serret coordinate system, based on the unit vectors

$$\vec{e}_{\rm tan} = \frac{d\vec{R}}{dl}\ ,\quad \vec{e}_{\rm norm} = -\left|\frac{d\vec{e}_{\rm tan}}{dl}\right|^{-1}\frac{d\vec{e}_{\rm tan}}{dl}\ ,\quad \vec{e}_{\rm bin} = \vec{e}_{\rm tan}\times\vec{e}_{\rm norm}\ . \tag{3}$$

In the here used coordinate system the last two vectors are rotated around the tangential vector [6, 2] by the torsion $T = \vec{e}_{\rm bin}\cdot\frac{d\vec{e}_{\rm norm}}{dl}$ of the reference curve,

$$\vec{e}_x + {\rm i}\vec{e}_y = e^{{\rm i}\vartheta}(\vec{e}_{\rm norm}+{\rm i}\vec{e}_{\rm bin})\ ,\quad \vartheta = \int_0^l T(l')\,dl'\ ,\quad \vec{e}_l = \vec{e}_{\rm tan}\ . \tag{4}$$

This coordinate system has two advantages. Firstly the torsion does not appear in the equation of motion and secondly the coordinate vectors given by the partial derivatives of a space point $\vec{r}(x,y,l) = \vec{R}(l) + x\vec{e}_x + y\vec{e}_y$ with respect to the coordinates x, y, and l are perpendicular to each other. Additionally we introduce a reference particle with momentum p_0, kinetic energy K_0 and time of flight t_0 which does not necessarily need to move on the reference curve $\vec{R}(l)$. For simplicity's sake however we use a reference particle which travels along the reference curve. Besides the distances from the reference curve, the relative deviation of the time and kinetic energy from the reference particle uniquely determine a point in the 6-dimensional phase space, $\vec{z} = (x,a,y,b,\tau,\delta)$,

$$x\ ,\quad a = \frac{p_{\rm x}}{p_0}\ ,\quad y\ ,\quad b = \frac{p_{\rm y}}{p_0}\ ,\quad \tau = \frac{K_0}{p_0}(t_0-t)\ ,\quad \delta = \frac{K-K_0}{K_0}\ . \tag{5}$$

It can be shown that these phase space coordinates are canonical variables [7]. In this coordinate system the equation of motion of the spin components in the coordinate system $\vec{S} = (S_{\rm x},S_{\rm y},S_{\rm l})^T$, $\vec{s} = S_{\rm x}\vec{e}_x + S_{\rm y}\vec{e}_y + S_{\rm l}\vec{e}_l$ assumes the following form:

$$\frac{d\vec{S}}{dl} = \vec{\Omega}\times\vec{S}\ ,\quad \vec{\Omega}(\vec{r},\vec{p},l) = \frac{hp}{v\,(\vec{p}\cdot\vec{e}_{\rm l})}\vec{\Omega}_{\rm BMT} - \vec{\kappa}\times\vec{e}_l \tag{6}$$

in which we make use of the curvature vector of the reference curve $\vec{\kappa} = (\kappa_{\rm x},\kappa_{\rm y},0)^T = -\frac{d}{dl}\vec{e}_{\rm tan}$ and $h = 1 + x\kappa_{\rm x} + y\kappa_{\rm y}$.

2 Solution of the spin orbit motion

The exact solution of the coupled spin orbit motion through an optical element is given by the transfer map $\vec{z}_f = \vec{M}(\vec{z}_i)$ and the 3×3 orthogonal spin transport matrix $\underline{A}(\vec{z}_i)$ determining the motion of an initial spin $\vec{S}_i$ to the final spin $\vec{S}_f = \underline{A}(\vec{z}_i)\vec{S}_i$. The rotation matrix $\underline{A}$ satisfies the differential equation

$$\frac{d}{dl}\underline{A}(\vec{z}_i, l) = \underline{\Omega}(\vec{z}_i, l)\,\underline{A}(\vec{z}_i, l) \ , \quad \underline{A}(\vec{z}_i, l = 0) = \underline{I}_{3\times3} \ , \quad \Omega_{ij} = -\varepsilon_{ijk}\Omega_k \ . \tag{7}$$

The matrix $\underline{\Omega}$ is a representation of the vector $\vec{\Omega}$ in equation (6) and depends on the phase space motion.

Rotations cannot only be presented by matrices of the $SO(3)$ group, but also by matrices of the $SU(2)$ group. This leads to Hamilton's quaternion representations. The vector $\vec{\gamma} = \vec{\beta}\sin\frac{\phi}{2}$ and the scalar $\kappa = \cos\frac{\phi}{2}$ uniquely describe a rotation by an angle ϕ around a direction $\vec{\beta}$. The connection between the two representations is known,

$$\vec{\gamma} = -\frac{1}{2\sqrt{\text{Sp}\,[\underline{A}] + 1}}\begin{pmatrix} A_{23} - A_{32} \\ A_{31} - A_{13} \\ A_{12} - A_{21} \end{pmatrix} \ , \quad \kappa = \frac{1}{2}\sqrt{\text{Sp}\,[\underline{A}] + 1} \ . \tag{8}$$

The calculation of the spin rotation in an accelerator turns out to be three times faster in the quaternion representation, justifying the additional theoretical expenses. The spin transport along two elements in matrix representation is, of course, the product of the two matrices. In the quaternion representation the whole quaternion of two elements (first through the element with index 1 and then through the element 2) is given by

$$\vec{\gamma}_{1/2} = \kappa_1\vec{\gamma}_2 + \kappa_2\vec{\gamma}_1 - \vec{\gamma}_1 \times \vec{\gamma}_2 \ , \quad \kappa_{1/2} = \kappa_1\kappa_2 - \vec{\gamma}_1 \cdot \vec{\gamma}_2 \tag{9}$$

We want to compute the spin transfer quaternion up to a given order in the initial phase space coordinates. The iteration of the spin rotation matrix $\underline{A}$ together with equation (8) leads to the desired iteration. However, it is more direct to use the equation of motion of the spin transfer quaternion [8, 9],

$$\frac{d\vec{\gamma}}{dl} = \frac{1}{2}\left(\vec{\Omega} \times \vec{\gamma} + \kappa\,\vec{\Omega}\right) \ , \quad \frac{d\kappa}{dl} = -\frac{1}{2}\vec{\gamma} \cdot \vec{\Omega} \tag{10}$$

directly for a successive approximation. By use of the four component vector $\vec{\gamma}_\kappa = (\vec{\gamma}, \kappa)$ the equation of motion (10) can be written in a more compact form,

$$\frac{d\vec{\gamma}_\kappa}{dl} = \underline{\Lambda}(\vec{z}_i, l)\vec{\gamma}_\kappa \ , \quad \vec{\gamma}_\kappa(l = 0) = \begin{pmatrix} \vec{0} \\ 1 \end{pmatrix} \tag{11}$$

where the matrix $\underline{\Lambda}$ is antisymmetric, $\underline{\Lambda}^T(\vec{z}_i, l) = -\underline{\Lambda}(\vec{z}_i, l)$. Therefore the length of the spin transfer quaternion remains unchanged, as required by $\vec{\gamma}^2 + \kappa^2 = 1$. The solution of this differential equation can be written by an orthogonal matrix $\underline{G}(\vec{z}_i, l)$,

$$\vec{\gamma}_\kappa = \underline{G}(\vec{z}_i, l)\begin{pmatrix} \vec{0} \\ 1 \end{pmatrix} \ . \tag{12}$$

Hence the matrix $\underline{G}(\vec{z}_i, l)$ satisfies the differential equation

$$\frac{d}{dl}\underline{G}(\vec{z}_i, l) = \underline{\Lambda}(\vec{z}_i, l)\underline{G}(\vec{z}_i, l) \ , \quad \underline{G}(\vec{z}_i, l = 0) = \underline{I}_{4\times4} \ . \tag{13}$$

3   Successive approximation of the spin orbit motion

The exact phase space motion $\vec{z} = \vec{M}(\vec{z_i})$ fulfills the equation of motion in the curvilinear coordinate system,

$$\frac{d}{dl}\vec{z}(l) = \vec{f}(\vec{z}(l),l) \ , \quad \vec{z}(0) = \vec{z_i} \ . \tag{14}$$

Although different procedures for the iterative calculation of the phase space motion are already known, our procedure used is shown for a better understanding of the spin computation. Namely all the approximations in use are based on the same proceedings which is demonstrated in detail by means of the phase space motion. First we expand the functions $\vec{M}(\vec{z_i})$ and $\vec{f}$ in a Taylor series with respect to the small initial phase space coordinates $\vec{z_i}$ about the origin in phase space,

$$\vec{z} = \sum_{k=0}^{\infty} \vec{z}^{(k)}(\vec{z_i},l) \ , \tag{15}$$

$$\vec{f}(\vec{z},l) = \vec{f}\left(\vec{z}^{(0)}(\vec{z_i},l),l\right) + \sum_{j=1}^{\infty}\vec{f}^{(j)}\left(\sum_{k=1}^{\infty}\vec{z}^{(k)}(\vec{z_i},l),l\right) \ . \tag{16}$$

The $\vec{z}^{(k)}(\vec{z_i},l)$ and $\vec{f}^{(j)}(\vec{z},l)$ express the order k and j in the initial coordinates and the vector $\vec{z}$ respectively. Then we insert the expansion in the equation of motion (14) and identify the particular orders by sorting both sides of (14) according to the same powers of the initial coordinates. By reason of our choice of the path of the reference particle the zero order vanishes, $\vec{z}^{(0)}(\vec{z_i},l) = \vec{0}$. The first order $\vec{z}^{(1)}(\vec{z_i},l)$ follows from the linearized equation of motion,

$$\frac{d}{dl}\vec{z}^{(1)}(\vec{z_i},l) = \vec{f}^{(1)}\left(\vec{z}^{(1)}(\vec{z_i},l),l\right) \ , \quad \vec{z}^{(1)}(\vec{z_i},l=0) = \vec{z_i} \ , \tag{17}$$

Next we reduce the differential equations of the higher orders to integrals of the lower ones. By inserting the first order $\vec{z}^{(1)}(\vec{z_i},l) = \underline{M}\vec{z_i}$ in (17) we acquire the key to the higher orders,

$$\vec{f}^{(1)}\left(\vec{z}^{(1)}(\vec{z_i},l),l\right) = \left[\frac{d}{dl}\underline{M}\right]\vec{z_i} = \left[\frac{d}{dl}\underline{M}\right]\underline{M}^{-1}\vec{z}^{(1)}(\vec{z_i},l) \ . \tag{18}$$

Since $\vec{f}^{(1)}\left(\vec{z}^{(1)}(\vec{z_i},l),l\right)$ is linear in $\vec{z}^{(1)}(\vec{z_i},l)$, eq. (18) is valid for $\vec{z}^{(n)}(\vec{z_i},l)$ as well,

$$\vec{f}^{(1)}\left(\vec{z}^{(n)}(\vec{z_i},l),l\right) = \left[\frac{d}{dl}\underline{M}\right]\underline{M}^{-1}\vec{z}^{(n)}(\vec{z_i},l) \ . \tag{19}$$

It is with this formula that we are now able to reduce the unpleasant differential equations of the higher orders ($n \geq 2$) to integrals of the lower, already calculated ones,

$$\frac{d}{dl}\vec{z}^{(n)}(\vec{z_i},l) - \vec{f}^{(1)}\left(\vec{z}^{(n)}(\vec{z_i},l),l\right) = \sum_{j=2}^{n}\left[\vec{f}^{(j)}\left(\sum_{k=1}^{n-1}\vec{z}^{(k)}(\vec{z_i},l),l\right)\right]_n \ , \tag{20}$$

$$\underline{M}(l)\frac{d}{dl}\left(\underline{M}^{-1}(l)\vec{z}^{(n)}(\vec{z_i},l)\right) = \sum_{j=2}^{n}\left[\vec{f}^{(j)}\left(\sum_{k=1}^{n-1}\vec{z}^{(k)}(\vec{z_i},l),l\right)\right]_n \ . \tag{21}$$

The brackets $[\dots]_n$ gives the homogeneous polynom of order n in the initial coordinates which the expression within the brackets contains. Multiplying both sides of (21) with the inverse of $\underline{M}(l)$ and integrating them we receive the successive approximation of the phase space motion,

$$\vec{z}^{\,(n)}\left(\vec{z}_{\mathrm{i}},l\right) = \underline{M}(l)\sum_{j=2}^{n}\int_{0}^{l}\underline{M}^{-1}(l')\left[\vec{f}^{\,(j)}\left(\sum_{k=1}^{n-1}\vec{z}^{\,(k)}\left(\vec{z}_{\mathrm{i}},l'\right),l'\right)\right]_{\mathrm{n}}dl' \, . \tag{22}$$

By analogy to the iteration of the phase space motion we expand the functions $\underline{G}$ and $\underline{\Lambda}$ in Taylor series,

$$\underline{G}(\vec{z}_{\mathrm{i}},l) = \sum_{k=0}^{\infty}\underline{G}^{(k)}(\vec{z}_{\mathrm{i}},l) \, , \quad \underline{\Lambda}(\vec{z}_{\mathrm{i}},l) = \sum_{j=0}^{\infty}\underline{\Lambda}^{(j)}(\vec{z}_{\mathrm{i}},l) \tag{23}$$

and insert the expansions in the equation of motion of $\underline{G}$,

$$\frac{d}{dl}\underline{G}(\vec{z}_{\mathrm{i}},l) = \underline{\Lambda}(\vec{z}_{\mathrm{i}},l)\underline{G}(\vec{z}_{\mathrm{i}},l) \, , \quad \underline{G}(\vec{z}_{\mathrm{i}},l=0) = \underline{I}_{4\times4} \, . \tag{24}$$

Subsequently we identify the particular orders by sorting both sides of (24) and find the zero given by

$$\frac{d}{dl}\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l) = \underline{\Lambda}^{(0)}(\vec{z}_{\mathrm{i}},l)\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l) \, , \quad \underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l=0) = \underline{I}_{4\times4} \, . \tag{25}$$

Finally we transform the nonlinear differential equation of the higher orders to integrals by use of (25),

$$\frac{d}{dl}\underline{G}^{(n)}(\vec{z}_{\mathrm{i}},l) \;=\; \sum_{k=0}^{n-1}\underline{\Lambda}^{(n-k)}(\vec{z}_{\mathrm{i}},l)\underline{G}^{(k)}(\vec{z}_{\mathrm{i}},l) + \underline{\Lambda}^{(0)}(\vec{z}_{\mathrm{i}},l)\underline{G}^{(n)}(\vec{z}_{\mathrm{i}},l) \tag{26}$$

$$\;=\; \sum_{k=0}^{n-1}\underline{\Lambda}^{(n-k)}(\vec{z}_{\mathrm{i}},l)\underline{G}^{(k)}(\vec{z}_{\mathrm{i}},l) + \left[\frac{d}{dl}\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l)\right]\left[\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l)\right]^{-1}\underline{G}^{(n)}(\vec{z}_{\mathrm{i}},l) \, , \tag{27}$$

$$\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l)\left[\frac{d}{dl}\left[\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l)\right]^{-1}\underline{G}^{(n)}(\vec{z}_{\mathrm{i}},l)\right] \;=\; \sum_{k=0}^{n-1}\underline{\Lambda}^{(n-k)}(\vec{z}_{\mathrm{i}},l)\underline{G}^{(k)}(\vec{z}_{\mathrm{i}},l) \, .$$

Remembering that $\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l)$ is orthogonal, we acquire from this equation the successive approximation of $\underline{G}$,

$$\underline{G}^{(n)}(\vec{z}_{\mathrm{i}},l) \;=\; \underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l)\sum_{k=0}^{n-1}\int_{0}^{l}\left[\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l')\right]^{T}\underline{\Lambda}^{(n-k)}(\vec{z}_{\mathrm{i}},l')\underline{G}^{(k)}(\vec{z}_{\mathrm{i}},l')\,dl' \, . \tag{28}$$

Multiplying (28) by the initial spin transfer quaternion $\vec{\gamma}_{\kappa}(\vec{z}_{\mathrm{i}},l=0) = (0,0,0,1)^{T}$, the desired iteration of $\vec{\gamma}_{\kappa}$ via its equation of motion is found,

$$\vec{\gamma}_{\kappa}^{\,(n)}(\vec{z}_{\mathrm{i}},l) \;=\; \underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l)\sum_{k=0}^{n-1}\int_{0}^{l}\left[\underline{G}^{(0)}(\vec{z}_{\mathrm{i}},l')\right]^{T}\underline{\Lambda}^{(n-k)}(\vec{z}_{\mathrm{i}},l')\vec{\gamma}_{\kappa}^{\,(k)}(\vec{z}_{\mathrm{i}},l')\,dl' \, . \tag{29}$$

For future use in simulation programs for HERA we analytically computed $\vec{\gamma}_{\kappa}$ up to the second order for the magnetic quadrupole, the magnetic dipole, for different combinations of quadrupole and dipole, and for the magnetic sextupole. We considered the fields in the

SCOFF-approximation. Nevertheless the algorithm can be applied to any electro-magnetic field. The results especially for the combinations of quadrupole and dipole are very lengthy. Therefore their presentation is restricted to only one example, the spin transfer scalar of the quadrupole, $\vec{B}_{\mathrm{quadr}}(x, y, l) = \omega_1^2 \frac{p_0}{q}(y\vec{e}_x + x\vec{e}_y)$ with $k_q = \omega_1^2$ as the quadrupole strength. In Table 1 the coefficient of the monomial $x^{k_x} a^{k_a} y^{k_y} b^{k_b} \tau^{k_\tau} \delta^{k_\delta}$ and the corresponding exponents $k_x k_a k_y k_b k_\tau k_\delta$ are given. We used the following abbreviations,

$$\mathsf{u} = \omega_1 l \ , \qquad \sin \mathsf{u} = \mathsf{su} \ , \qquad \sinh \mathsf{u} = \mathsf{shu} \ ,$$

$$\sin \frac{\mathsf{u}}{2} = \mathsf{suh} \ , \quad \sinh \frac{\mathsf{u}}{2} = \mathsf{shuh} \ , \qquad \chi = 1 + a\gamma_0 \ .$$

The second order expansions of the other elements are listed in [9].

κ	$k_x k_a k_y k_b k_\tau k_\delta$
1	000000
$-\frac{1}{8}\chi^2 \omega_1^2 \, \mathsf{su}^{\,2}$	200000
$-\frac{1}{2}\chi^2 \omega_1 \, \mathsf{su} \, \mathsf{suh}^{\,2}$	110000
$-\frac{1}{2}\chi^2 \, \mathsf{suh}^{\,4}$	020000
$-\frac{1}{8}\chi^2 \omega_1^2 \, \mathsf{shu}^{\,2}$	002000
$-\frac{1}{2}\chi^2 \omega_1 \, \mathsf{shu} \, \mathsf{shuh}^{\,2}$	001100
$-\frac{1}{2}\chi^2 \, \mathsf{shuh}^{\,4}$	000200

Table 1: The scalar κ of $\vec{\gamma}_\kappa$ for the magnetic quadrupole

4 Outlook

In the next stage the results will be worked into SPRINT [10, 3, 4], the main simulation program for spin motion in HERA. In detail the following topics should be discussed:

- Influence of the second-order spin motion on the invariant spin field near resonances

- Nonlinear influence of the beam-beam interaction

- Nonlinear influence of the sextupoles which correct the chromaticity and the tune spread

References

[1] K. L. Brown, *Report SLAC* **75**, 70–134 (1982)

[2] G. H. Hoffstaetter, *Proceedings PAC* **94**, Dallas/TX (1995)

[3] G. H. Hoffstaetter, *In this book, DESY* , Hamburg (1999)

[4] M. Vogt, *In this book, DESY* , Hamburg (1999)

[5] V. Bargmann, L. Michel, V. L. Telegdi, *Phys. Rev. Lett.* **2**, 435-436 (1959)

[6] H. Rose, *Nucl. Instrum. Methods* **A258**, 374-401 (1987)

[7] E. D. Courant and H. S. Snyder, *Annals of physics* **3**, 1–48 (1958)

[8] G. H. Hoffstaetter, Ultramicroscopy , submitted (1999)

[9] C. Weissbaecker, TU-Darmstadt Diplomarbeit, DESY-Thesis **1999-019**, Hamburg (1999)

[10] G. H. Hoffstaetter, *Phys. Rev.* E **54**, 4240-4255 (1996)