Accelerator Rings with Polarized Beams and Spin Manipulation *

Georg H. Hoffstaetter
Cornell University, Ithaca, New York

ABSTRACT

The basic formulas of describing polarization dynamics in accelerators will be presented. These include the equation of spin motion in a comoving coordinate frame for spin vectors, spin transport matrices, spin transport quaternions, and spinors. It will also be shown how spin fields evolve in these four ways of descibing spin motion. Furthermore, some basic concepts of polarized beams in ring accelerators will be discussed. These include the periodic spin direction on the closed orbit, the closed orbit spin tune, the invariant spin field, the amplitude dependent spin tune, the Froissart Stora formula, and transfer maps of linearized spin-orbit motion, with their relation to resonance strength and to spin-orbit coupling integrals. The here presented material is intended to be a tutorial of the basic processes involved when polarized beams travel around circular accelerators and of the theory used to describe first order effects. Much of this material has been presented in [1], where it served as the basis to extend the analysis of spin motion to higher–order phenomena.

1 Introduction

While electron beams in high energy accelerators can polarize automatically due to the emission of spin-flip synchrotron radiation via the Sokolov-Ternov effect, electrons of lower energy and protons have to be polarized in a source. These beams then have to be accelerated and stored with little loss of polarization. We will here not be concerned with the spin-flip synchrotron radiation but with the radiation free dynamics of pre-polarized beams. Today polarized proton beams can be produced either by a polarized atomic beam source (ABS) or in an optically pumped polarized ion source (OPPIS). Pulsed beams with polarization of up to 87% for 1mA H^- beam current [3] and up to 60% for 5mA [4], respectively, have been achieved with these sources.

After a review in section 2.1 of the various ways of formulating spin motion the concept of an invariant spin field will be used to understand features of acceleration and storage of polarized beams. The beam average of this field describes the maximum polarization available for particle physics experiments during the storage time of several hours. Furthermore, it allows first–order and higher–order resonances to be analyzed [5]. Crossing these resonances while accelerating the beam can lead to a reduction of polarization. While the invariant spin field and the amplitude dependent spin tune can be used to compute higher–order resonance strength, we will here only use them in the first-order treatment of linearized spin-orbit motion, and thus relate them to spin-orbit coupling integrals and to first order resonance strength, which in tern are regularly used in the Froissart Stora Fromula. Together, this amplifies the interdependence of these first order methods.

The invariant spin field to analyze spin dynamics at high proton energies has become the basis of a very detailed analysis of the acceleration process in HERA–p [1, 6] and is also becoming adopted by the RHIC group [7] for simulations of polarized beam in the AGS and of their planned 250GeV polarized proton beam. While these methods are novel and powerful, they go bejond first order and will not be covered here.

*Lecture for the COSY Summer School 2002

2 Spin Dynamics

2.1 The Equation of Spin Motion

The expectation value of the vector operator representing the spin of a particle satisfies the equation of motion of a classical spin vector. Viewed in the particle's instantaneous rest frame, the direction of this expectation value will here be denoted by *the spin* $\vec{s}$ with $|\vec{s}| = 1$. This direction is $\frac{2}{\hbar}$ times the expectation value. The polarization P of a beam is defined as the absolute value of the spin average taken over all N particles of the beam,

$$P = |\frac{1}{N}\sum_{j=1}^{N}\vec{s}_j| = |<\vec{s}>_N| \ . \tag{1}$$

The expectation value $\vec{s}$ changes with the time t of the laboratory frame according to the Thomas–Bargmann–Michel–Telegdi (T-BMT) equation [8, 9]

$$\frac{d}{dt}\vec{s} = \vec{\Omega}_{BMT}(\vec{r},\vec{p}) \times \vec{s} \ . \tag{2}$$

The precession vector $\vec{\Omega}_{BMT}(\vec{r},\vec{p})$ depends on the particle's position $\vec{r}$ and its momentum $\vec{p}$. It can be expressed by the electric and magnetic fields $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$, by the particle's charge q and its rest energy mc^2, the relativistic factor γ, and by the particle's anomalous gyro-magnetic g-factor $G = (g-2)/2$ in the following way:

$$\vec{\Omega}_{BMT}(\vec{r},\vec{p}) = -\frac{q}{m}[(\frac{1}{\gamma}+G)\vec{B} - \frac{G\vec{p}\cdot\vec{B}}{\gamma(\gamma+1)m^2c^2}\vec{p} - \frac{1}{mc^2\gamma}(G+\frac{1}{1+\gamma})\vec{p}\times\vec{E}] \ . \tag{3}$$

All frame dependent quantities are taken in the laboratory frame. The anomalous g-factor is about 1.793 for protons, about 0.00116 for electrons, and about -0.143 for deuterons.

2.1.1 Spin motion in Flat Circular Accelerators

When introducing the components of the magnetic field $\vec{B}_\perp$ and $\vec{B}_\parallel$ which are perpendicular and parallel to the particle's momentum, the Lorentz force equation and the T-BMT equation in purely magnetic fields show some similarities,

$$\frac{d}{dt}\vec{p} = -\frac{q}{m\gamma}\{\qquad \vec{B}_\perp \qquad\}\times\vec{p} \ , \tag{4}$$

$$\frac{d}{dt}\vec{s} = -\frac{q}{m\gamma}\{(G\gamma+1)\vec{B}_\perp + (1+G)\vec{B}_\parallel\}\times\vec{s} \ . \tag{5}$$

In a solenoid magnet, $\vec{B}_\parallel$ produces a spin rotation around the longitudinal direction. The rotation angle is given by $d\phi = -(1+G)\frac{qB_\parallel}{p}dl$. For the circular accelerators HERA and COSY, for example, this leads to the following rotations:

- For 920GeV protons a spin rotation around the longitudinal of π is created by a longitudinal magnetic field integral of 3456Tm. For deuterons 11250Tm would even be needed, making solenoids completely impractical at such high energies. For HERA's 27.5GeV electron beam, 288Tm would be needed. Protons and deuterons with up to 3.3GeV/c in the COSY synchrotron would however require only 12.39Tm and 40.35Tm respectively to be rotated by π.

- Another interesting property to observe is that a particle's plane of focusing is rotated in a solenoid by an angle $d\varphi = -\frac{1}{2}\frac{qB_\parallel}{p}dl$ so that the spin rotation and the orbit rotation are related by the gyromagnetic factor: $d\phi = gd\varphi$.

Also in transverse magnetic fields, where $\vec{B}_{\|} = 0$, several conclusions can immediately be drawn from these equations. This case is especially relevant, since a flat circular accelerator has only vertical magnetic fields in the center plane.

- In such a transverse magnetic field, the momentum $\vec{p}$ rotates in the plane perpendicular to the field. If $\vec{s}_p$ describes the spin in a coordinate system which rotates with the particle's momentum, the equation of spin motion relative to the particle motion becomes $\frac{d}{dt}\vec{s}_p = -\frac{q}{m}G\vec{B}_{\perp} \times \vec{s}_p$. The spin rotation relative to the orbit motion is therefore independent of energy, in contrast to the orbit deflection which varies like $1/\gamma$. For fixed orbit deflections and thus fixed ratio of $\vec{B}_{\perp}/\gamma$, the spin precession rate, however, increases with energy. For protons with velocity v close to the speed of light, a fixed field integral of $\int Bdl = \pi\frac{mcv}{qG} \approx 5.48\text{Tm}$ leads to a spin rotation of π. Electrons require a field integral of 4.62Tm for this rotation angle, and deuterons require a field integral of 137.2Tm. Deuteron spins are therefore about 25 times harder to manipulate with magnetic fields than proton spins. On the other hand a deuteron's spin direction is about 25 times less sensitive to field errors.

- If the orbit is deflected by an angle ϕ in a transverse magnetic field, then the spin is rotated by an angle $G\gamma\phi$ relative to the orbit. To create a spin rotation of $\pi/2$, the orbit for 920GeV protons only has to change by 0.89mrad, for 920GeV deuterons by -22.1mrad, and for 27.5GeV Electrons by 24.8mrad. The minus sign indicates that deuterons rotate in the oposite direction. For protons or deuterons with 3.3GeV/c in the COSY synchrotron the required orbit devlection is $13.7^\circ = 240\text{mrad}$ and -311°.

- In a flat ring, the orbit deflection angle of 2π during one turn leads to $G\gamma$ full spin rotations around the vertical direction relative to the particle's direction. For 920GeV these are 1756 such rotations. This number of spin rotations performed during one turn along the closed orbit is called the closed-orbit spin tune ν_0 . A 27.5GeV electron beam in HERA-e has $\nu_0 = 62.5$. Deuterons with 920GeV would make -70 rotations. Protons and deuterons in the COSY synchrotron with up to 3.3GeV/c rotate 6.54 and -0.29 times.

- Whenever the energy of a proton is increased by 523MeV, the spin rotates once more per revolution around the ring. For an electron, this energy increase is 441MeV, and for a deuteron it is 13.1GeV.

2.1.2 Spin Motion in the Curvilinear Coordinate System

The design trajectory of a particle accelerator is described by a space curve $\vec{R}(l)$ with $|d\vec{R}(l)| = dl$. A coordinate system is defined relative to this curve with the second unit vector tangential to the curve and the first and third unit vectors chosen to obtain a right handed orthonormal set of vectors called an orthonormal dreibein. The first and third unit vectors therefore lie in a plane perpendicular to the curve. The orientation of the unit vectors in that plane is arbitrary and can change along the curve. The Frenet-Serret coordinate system is defined by

$$\vec{t}_2 = \frac{d}{dl}\vec{R}(l) \ , \quad \frac{1}{\rho} = |\frac{d}{dl}\vec{t}_2| \ , \quad \vec{t}_1 = -\rho\frac{d}{dl}\vec{t}_2 \ , \quad \vec{t}_3 = \vec{t}_1 \times \vec{t}_2 \ , \quad T = -\vec{t}_3 \cdot \frac{d}{dl}\vec{t}_1 \ , \tag{6}$$

where the torsion of the space curve $\vec{R}(l)$ is given by T . From these definitions and with $\frac{d}{dl}(\vec{t}_1 \cdot \vec{t}_2) = \vec{t}_2 \cdot \frac{d}{dl}\vec{t}_1 - \frac{1}{\rho} = 0$ it follows that

$$\frac{d}{dl}\vec{t}_1 = -T\vec{t}_3 + \frac{1}{\rho}\vec{t}_2 \ , \quad \frac{d}{dl}\vec{t}_3 = -\vec{t}_2 \times \frac{d}{dl}\vec{t}_1 = T\vec{t}_1 \ . \tag{7}$$

A space vector $\vec{r}$ is specified by l and by the two coordinates x and y via

$$\vec{r} = \vec{R}(l) + x\vec{t}_1(l) + y\vec{t}_3(l) \ . \tag{8}$$

A space curve is then specified by the two functions $x(l)$ and $y(l)$. The derivative with respect to l of such a space curve $\vec{r}(l)$ is given by

$$\frac{d}{dl}\vec{r}(l) = (\frac{d}{dl}x + Ty)\vec{t}_1 + (\frac{d}{dl}y - Tx)\vec{t}_3 + (1 + \frac{x}{\rho})\vec{t}_2 \ . \tag{9}$$

To remove the torsion T from the equation of motion, one introduces the unit coordinate vectors $\vec{e}_x$, $\vec{e}_l$, $\vec{e}_y$ by winding back the rotation which is due to the torsion,

$$\vartheta = \int_{l_0}^{l} T(\tilde{l})d\tilde{l} \ , \quad \vec{e}_x + i\vec{e}_y = e^{i\vartheta}(\vec{t}_1 - i\vec{t}_3) \quad \vec{e}_l = \vec{t}_2 \ . \tag{10}$$

The coordinate system with the unit vectors $\vec{e}_x$, $\vec{e}_l$, $\vec{e}_y$ is shown in figure 1 and is called the curvilinear coordinate system. It follows that

$$\frac{d}{dl}\vec{e}_x + i\frac{d}{dl}\vec{e}_y = e^{i\vartheta}\{-T\vec{t}_3 + \frac{1}{\rho}\vec{t}_2 - iT\vec{t}_1 + iT(\vec{t}_1 - i\vec{t}_3)\} = \frac{e^{i\vartheta}}{\rho}\vec{e}_l \ . \tag{11}$$

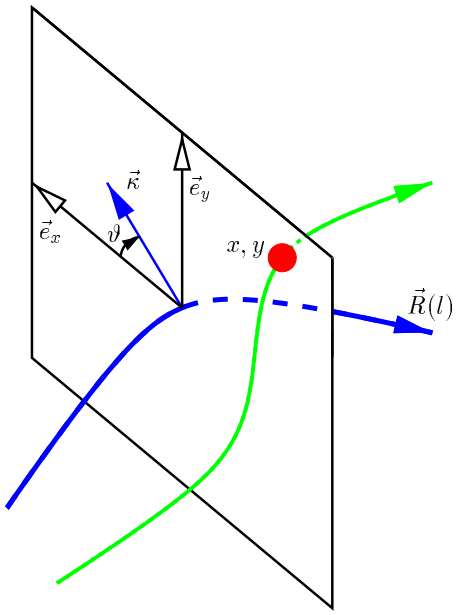

Figure 1: The unit vectors $\vec{e}_x$ and $\vec{e}_x$, the curvature vector $\vec{\kappa}$ of the design curve $\vec{R}(l)$ and the generalized coordinates x, y, and l of the curvilinear coordinate system. This system is rotated by θ with respect to the Frenet-Serret coordinate system.

For the right handed orthonormal dreibein $[\vec{e}_x, \vec{e}_l, \vec{e}_y]$ of the curvilinear coordinate system [10, 11], one obtains

$$\frac{d}{dl}\vec{R}(l) = \vec{e}_l \ , \quad \vec{r} = \vec{R}(l) + x\vec{e}_x + y\vec{e}_y \ , \tag{12}$$

$$\frac{d}{dl}\vec{e}_x = \frac{\cos\vartheta}{\rho}\vec{e}_l \ , \quad \frac{d}{dl}\vec{e}_y = \frac{\sin\vartheta}{\rho}\vec{e}_l \ , \tag{13}$$

$$\frac{d}{dl}\vec{e}_l = -\frac{1}{\rho}\vec{t}_1 = -\frac{1}{\rho}(\cos\vartheta\vec{e}_x + \sin\vartheta\vec{e}_y) \ , \tag{14}$$

$$\frac{d}{dl}\vec{r} = \vec{e}_x\frac{d}{dl}x + \vec{e}_y\frac{d}{dl}y + (1 + \frac{x\cos\vartheta + y\sin\vartheta}{\rho})\vec{e}_l \ . \tag{15}$$

For ease of notation, one can use $\vec{x} = (x, y)^T$, $\vec{\kappa} = (\cos\vartheta, \sin\vartheta)^T/\rho$, and $h = 1 + \vec{x}\cdot\vec{\kappa}$. Vectors like $\vec{p}$ which have a component in the $\vec{e}_3$ direction are described by

$$\vec{p} = p_x\vec{e}_x + p_y\vec{e}_y + p_l\vec{e}_l \tag{16}$$

$$\frac{d}{dl}\vec{p} = (\frac{d}{dl}p_x - p_l\kappa_x)\vec{e}_x + (\frac{d}{dl}p_y - p_l\kappa_y)\vec{e}_y + (\frac{d}{dl}p_l + p_x\kappa_x + p_y\kappa_y)\vec{e}_l \ . \tag{17}$$

To find the equations of particle motion in the curvilinear coordinate system, the independent coordinates in the equations of motion is changed from time t to arc length l by using

$$\frac{dt}{dl} = (\vec{e}_l\cdot\frac{d}{dl}\vec{r})/(\vec{e}_l\cdot\frac{d}{dt}\vec{r}) = \frac{h}{v}\frac{p}{p_l} \ , \tag{18}$$

where v is the velocity and $p = |\vec{p}|$ is the momentum. Properties of a reference particle moving on the design trajectory are We indicated by subscripts 0 and define the coordinates of all other particles relative to this reference particle through

$$x \ , \quad a = \frac{p_x}{p_0} \ , \quad y \ , \quad b = \frac{p_y}{p_0} \ , \quad \tau = (t_0 - t)\frac{K_0}{p_0} \ , \quad \delta = \frac{K - K_0}{K_0} \ . \tag{19}$$

where $K = mc^2(\gamma - 1)$ is the kinetic energy. These six phase space variables are denoted by the phase space vector $\vec{z}$. The coordinate pairs (x, a), (y, b), and (τ, δ) are canonically conjugate. Since $\vec{R}(l)$ is the path of the reference particle, the particle transport is origin preserving, because a particle with $\vec{z} = 0$ will continue to travel along the design trajectory. The equation of motion for these phase space coordinates with l as independent variable [11] is obtained by transforming the Lorentz force equation. Here I neglect the Stern–Gerlach forces since they are very small in comparison with the Lorentz force.

To transform the equations of spin motion into the curvilinear coordinate system, $\frac{d}{dl}t = \frac{h}{v}\frac{p}{p_l}$ from equation (18) is used. The spin direction $\vec{s}$ is expressed by its components in the curvilinear coordinate system and the column vector of these components is written as $\vec{S}$. A potential torsion of the reference curve does not enter the equations of particle motion in this coordinate system and it also does not enter the equation of spin motion,

$$\vec{s} = S_x\vec{e}_x + S_y\vec{e}_y + S_l\vec{e}_l \ , \tag{20}$$

$$\frac{d}{dl}\vec{s} = (\frac{d}{dl}S_x - S_l\kappa_x)\vec{e}_x + (\frac{d}{dl}S_y - S_l\kappa_y)\vec{e}_y + (\frac{d}{dl}S_l + S_x\kappa_x + S_y\kappa_y)\vec{e}_l$$
$$= \frac{h}{v}\frac{p}{p_l}\vec{\Omega}_{BMT}(\vec{r}, \vec{p}) \times \vec{s} \ . \tag{21}$$

For the column vector $\vec{S}$, the equation of motion is therefore given by

$$\frac{d}{dl}\vec{S} = \{\vec{\Omega}_{BMT}(\vec{r}, \vec{p})\frac{hp}{vp_l} - \vec{\kappa}\times\vec{e}_l\}\times\vec{S} \ . \tag{22}$$

The precession vector depends on the position and the momentum. This can be expressed as a dependence on l and on the 6 dimensional phase space variable $\vec{z}$.

2.1.3 Equation of Motion for Spins and Spin Fields

In a circular accelerator with circumference L, it is convenient to choose the azimuth $\theta = 2\pi l/L$ as independent variable, rather than the arc length l of the design trajectory. The coordinate vectors are not changed, with $\vec{e}_\theta = \vec{e}_l$. All fields are then 2π periodic in θ. The equation of particle motion is therefore 2π periodic,

$$\frac{d}{d\theta}\vec{z} = \vec{v}(\vec{z}, \theta) \ , \quad \vec{v}(\vec{z}, \theta + 2\pi) = \vec{v}(\vec{z}, \theta) \ , \tag{23}$$

$$\frac{d}{d\theta}\vec{S} = \vec{\Omega}(\vec{z}, \theta)\times\vec{S} \ , \quad \vec{\Omega}(\vec{z}, \theta + 2\pi) = \vec{\Omega}(\vec{z}, \theta) \ , \tag{24}$$

where the precession vector is obtained from equation (22) as

$$\vec{\Omega}(\vec{z},\theta) = \frac{L}{2\pi}(\vec{\Omega}_{BMT}(\vec{r},\vec{p})\frac{hp}{vp_l} - \vec{\kappa}\times\vec{e}_l) \ . \tag{25}$$

A particle starting with an initial phase space coordinate $\vec{z}_i$ and with an initial spin $\vec{S}_i$ propagates around an accelerator according to the equations of spin-orbit motion (24). After it has traveled from azimuth θ_0 to θ, it will have the coordinates $\vec{z}(\theta) = \vec{M}(\vec{z}_i,\theta_0;\theta)$ and $\vec{S}(\theta) = \underline{R}(\vec{z}_i,\theta_0;\theta)\vec{S}_i$, where $\vec{M}(\vec{z}_i,\theta_0;\theta)$ is called the transport map and the orthogonal matrix $\underline{R}(\vec{z}_i,\theta_0;\theta)$ is called the spin transport matrix.

This rotation matrix can be computed by tracking three linearly independent spins along the phase space trajectory starting with $\vec{z}_i$ at azimuth θ_0. Transporting the nine real coefficients of these vectors is however not an efficient way of simulating spin motion, since a rotation can be described by three real numbers. Furthermore, the orthogonal structure of $\underline{R}$ does not change the angle between two spins which travel along the same trajectory and it does not change the length of a spin. These properties can be violated either by numerical errors or by computational approximations when individual spins are propagated. Therefore, more efficient methods will be introduced below.

A particle beam consists of particles at different phase space positions. Each particle can have a different spin direction. The function $\vec{f}(\vec{z},\theta)$ describing the spin direction for a particle at phase space point $\vec{z}$ at azimuth θ is called a spin field. The equation of motion for a spin field is thus given by

$$\frac{d}{d\theta}\vec{f} = \partial_\theta\vec{f} + [\vec{v}(\vec{z},\theta)\cdot\partial_{\vec{z}}]\vec{f} = \vec{\Omega}(\vec{z},\theta)\times\vec{f} \ . \tag{26}$$

2.1.4 Equation of Motion for the Spin Transport Matrix

In the following sections I will investigate various methods for describing the propagation of spins and spin fields along particle trajectories. Inserting the relation $\vec{S}(\theta) = \underline{R}(\vec{z}_i,\theta_0;\theta)\vec{S}_i$ into the equation of motion (24) leads to the equation of motion for the spin transport matrix

$$\partial_\theta\underline{R}(\vec{z}_i,\theta_0;\theta) = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}\underline{R}(\vec{z}_i,\theta_0;\theta) \ , \quad \underline{R}(\vec{z}_i,\theta_0;\theta_0) = \underline{1}_3 \ , \tag{27}$$

where $\underline{1}_3$ describes the 3×3 dimensional unit matrix. The spin rotation matrix for a particle trajectory which enters the nth particle optical element with $\vec{z}_{n-1}$ is computed by multiplying the spin transport matrices $\underline{R}_n(\vec{z}_{n-1})$ of the individual elements. This method has the same disadvantage as the transport of three individual spins. Nine real coefficients are transported, where three could already describe a rotation. Furthermore, computational inaccuracies can again lead to violations of the orthogonal structure of the matrix, which therefore has to be orthogonalized whenever such violations become problematic.

Using the transport matrix, a spin is propagated by $\vec{S}(\theta) = \underline{R}(\vec{z}_i,\theta_0;\theta)\vec{S}_i$ and a spin field $\vec{f}(\vec{z},\theta)$ can be propagated by

$$\vec{f}(\vec{z},\theta) = \underline{R}(\vec{z}_i,\theta_0;\theta)\vec{f}(\vec{z}_i,\theta_0) \ \text{ with } \ \vec{z}_i = \vec{M}(\vec{z},\theta;\theta_0) \ . \tag{28}$$

Here the inverse transport map $\vec{M}(\vec{z},\theta;\theta_0) = \vec{M}^{-1}(\vec{z},\theta_0;\theta)$ describing the reverse motion from θ back to θ_0 has been used.

2.1.5 Equation of Motion for the Spin Transport Quaternion

As will now be demonstrated, it is more efficient to use an SU(2) representation rather than the SO(3) matrices when describing the rotations of spins. The matrix $\underline{R}$ of equation (27) describes

the rotation of an initial spin $\vec{S}_i$ around a unit rotation vector $\vec{e}$ by an angle α. Splitting the spin into components parallel and perpendicular to $\vec{e}$, one obtains

$$\vec{S}(\theta) = \vec{e}(\vec{S}_i \cdot \vec{e}) + \cos\alpha[\vec{S} - \vec{e}(\vec{S}_i \cdot \vec{e})] + \sin\alpha\, \vec{e} \times \vec{S}_i \ . \tag{29}$$

With $a_0 = \cos\frac{\alpha}{2}$ and $\vec{a} = \sin\frac{\alpha}{2}\vec{e}$, the matrix $\underline{R}$ can therefore be written as [13]

$$R_{ij} = (a_0^2 - \vec{a}^2)\delta_{ij} + 2a_i a_j - 2a_0\epsilon_{ijk}a_k \ , \tag{30}$$

where the vector product is expressed using the totally antisymmetric tensor ϵ_{ijk}. The SU(2) matrix representing a rotation around $\vec{e}$ by the angle α is given by the quaternion

$$A = \exp(-i\frac{\alpha}{2}\vec{e}\cdot\underline{\vec{\sigma}}) = a_0\underline{1}_2 - i\vec{a}\cdot\underline{\vec{\sigma}} \ . \tag{31}$$

Here the elements of the vector $\underline{\vec{\sigma}}$ are the three Pauli matrices. If a particle traverses an optical element which rotates the spin according to the quaternion A and then passes through an element which rotates the spin according to the quaternion B, the total rotation of the spin is given by

$$\begin{aligned} C &= c_0\underline{1}_2 - i\vec{c}\cdot\underline{\vec{\sigma}} = (b_0\underline{1}_2 - i\vec{b}\cdot\underline{\vec{\sigma}})(a_0\underline{1}_2 - i\vec{a}\cdot\underline{\vec{\sigma}}) \\ &= (b_0a_0 - \vec{b}\cdot\vec{a})\underline{1}_2 - i(b_0\vec{a} + \vec{b}a_0 + \vec{b}\times\vec{a})\cdot\underline{\vec{\sigma}} \ . \end{aligned} \tag{32}$$

This concatenation of quaternions can be written in matrix form as

$$\vec{C} = \begin{pmatrix} c_0 \\ \vec{c} \end{pmatrix} = \underline{B}\begin{pmatrix} a_0 \\ \vec{a} \end{pmatrix} \ , \quad \underline{B} = \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & b_0 & -b_3 & b_2 \\ b_2 & b_3 & b_0 & -b_1 \\ b_3 & -b_2 & b_1 & b_0 \end{pmatrix} \ . \tag{33}$$

Sometimes it is useful to have the quaternions appear in reversed order, even though particles travel first through the optical element corresponding to A,

$$\vec{C} = \begin{pmatrix} c_0 \\ \vec{c} \end{pmatrix} = \underline{\tilde{A}}\begin{pmatrix} b_0 \\ \vec{b} \end{pmatrix} \ , \quad \underline{\tilde{A}} = \begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{pmatrix} \ . \tag{34}$$

Since any quaternion vector has unit length, the matrices $\underline{B}$ and $\underline{\tilde{A}}$ are both orthogonal.

It has turned out to be useful to represent rotations in terms of a_0 and $\vec{a}$ for the following three reasons:

1. only 4 components are needed to describe and concatenate the rotation of spins,
2. even when numerical inaccuracies cause a small error in the computation of this representation, one can always normalize so that $a_0^2+\vec{a}^2 = 1$, which then always leads to a an orthogonal spin transport matrix,
3. only 28 floating point operations are required to compute the combined spin transport quaternion of two particle optical elements from their individual quaternions. The multiplication of the spin transport matrices requires 45 floating point operations.

While particles are propagating along the design curve by a distance $d\theta$, spins are rotated by an angle $|\vec{\Omega}|d\theta$ around the vector $\vec{\Omega}$. After having been propagated to θ by the quaternion A, a spin gets propagated from θ to $\theta + d\theta$ by the quaternion B with $b_0 = 1$ and $\vec{b} = \frac{1}{2}\vec{\Omega}d\theta$. The resulting total rotation is given by $A + d\theta\frac{d}{d\theta}A$ and one obtains the differential equation

$$\frac{d}{d\theta}\begin{pmatrix} a_0 \\ \vec{a} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\Omega_1 & -\Omega_2 & -\Omega_3 \\ \Omega_1 & 0 & -\Omega_3 & \Omega_2 \\ \Omega_2 & \Omega_3 & 0 & -\Omega_1 \\ \Omega_3 & -\Omega_2 & \Omega_1 & 0 \end{pmatrix}\begin{pmatrix} a_0 \\ \vec{a} \end{pmatrix} \ . \tag{35}$$

Writing the vector as $\vec{A}$ and the matrix as $\underline{\Omega}$, the spin-orbit equation of motion takes the form

$$\frac{d}{d\theta}\vec{z} = \vec{v}(\vec{z},\theta) \ , \quad \frac{d}{d\theta}\vec{A} = \frac{1}{2}\underline{\Omega}(\vec{z},\theta)\vec{A} \ . \tag{36}$$

The starting conditions at the initial azimuth θ_0 are $\vec{z} = \vec{z}_i$, $a_0 = 1$, and $\vec{a} = 0$. Sometimes, an equation of motion for the quaternion A itself is used rather than for the component vector $\vec{A}$,

$$\frac{d}{d\theta}A = -i\frac{1}{2}\vec{\Omega}\cdot\underline{\vec{\sigma}}A \ , \tag{37}$$

with the starting condition $A = \underline{1}_2$. When $\vec{A}(\vec{z}_i,\theta_0;\theta)$ is known, $\underline{R}(\vec{z}_i,\theta_0;\theta)$ can be constructed using equation (30) and one can again propagate an initial spin $\vec{S}_i$ and a spin field $\vec{f}(\vec{z}_i,\theta_0)$ by equation (28).

2.1.6 Equation of Motion for Spinors

In the SU(2) representation of rotations, a spin $\vec{S}$ is written in terms of the spinor $\Psi = (\psi_1,\psi_2)^T$ as $\vec{S} = \Psi^\dagger\underline{\vec{\sigma}}\Psi$ where ψ_1 and ψ_2 are two complex numbers. To have $|\vec{S}| = 1$, it is required that $|\psi_1|^2 + |\psi_2|^2 = 1$. The spinor represents a spin direction in polar coordinates ϑ and ϕ, which is illustrated by the fact that the following spinor and the following vector describe the same spin:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\vartheta}{2}e^{i\phi_1} \\ \sin\frac{\vartheta}{2}e^{i\phi_2} \end{pmatrix} \implies \vec{S} = \begin{pmatrix} \sin\vartheta\cos(\phi_2-\phi_1) \\ \sin\vartheta\sin(\phi_2-\phi_1) \\ \cos\vartheta \end{pmatrix} \ . \tag{38}$$

The equation of motion for the spinor is given by

$$\frac{d}{d\theta}\Psi = -i\frac{1}{2}(\vec{\Omega}\cdot\underline{\vec{\sigma}})\Psi \ , \tag{39}$$

which leads back to the vector form of the differential equation of spin motion [14],

$$\begin{aligned} \frac{d}{d\theta}\vec{S} &= (\frac{d}{d\theta}\Psi^\dagger)\underline{\vec{\sigma}}\Psi + \Psi^\dagger\underline{\vec{\sigma}}(\frac{d}{d\theta}\Psi) \\ &= i\frac{1}{2}\Psi^\dagger[(\vec{\Omega}\cdot\underline{\vec{\sigma}})\underline{\vec{\sigma}} - \underline{\vec{\sigma}}(\vec{\Omega}\cdot\underline{\vec{\sigma}})]\Psi = \Psi^\dagger[\vec{\Omega}\times\underline{\vec{\sigma}}]\Psi = \vec{\Omega}\times\vec{S} \ . \end{aligned} \tag{40}$$

If a spin rotates by an angle α around a θ independent unit rotation vector $\vec{e}$ while the particle travels to θ, then equation (39) leads to the spinor propagation relation $\Psi(\theta) = \exp(-i\frac{\alpha}{2}\vec{e}\cdot\underline{\vec{\sigma}})\Psi_i$. A spinor is therefore propagated through an accelerator by the spin transport quaternion of equation (31),

$$\Psi(\theta) = (a_0\underline{1}_2 - i\vec{a}\cdot\underline{\vec{\sigma}})\Psi_i \ . \tag{41}$$

If a spin is parallel to the rotation vector $\vec{e}$, it is not changed during the rotation. The corresponding spinor Ψ_e however is changed by a phase factor. To show this, the polar coordinates ϑ and ϕ of the vector $\vec{e}$ are used and the free phase of the spinor is indicated by $e^{i\xi}$,

$$\Psi_e = e^{i\xi}\begin{pmatrix} \cos\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2}e^{i\phi} \end{pmatrix} \ , \tag{42}$$

$$\begin{aligned} \Psi(\theta) &= \exp(-i\frac{\alpha}{2}\vec{e}\cdot\underline{\vec{\sigma}})\Psi_i = (\cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\underline{\vec{\sigma}}\cdot\vec{e})\Psi_i \\ &= \begin{pmatrix} \cos\frac{\alpha}{2} - i\sin\frac{\alpha}{2}\cos\vartheta & -ie^{-i\phi}\sin\frac{\alpha}{2}\sin\vartheta \\ -ie^{i\phi}\sin\frac{\alpha}{2}\sin\vartheta & \cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\cos\vartheta \end{pmatrix}\begin{pmatrix} \cos\frac{\vartheta}{2} \\ \sin\frac{\vartheta}{2}e^{i\phi} \end{pmatrix}e^{i\xi} = e^{-i\frac{\alpha}{2}}\Psi_e \end{aligned} \tag{43}$$

In the spinor formalism, the phase change of the spinor which describes the rotation vector can therefore be used to determine the rotation angle α.

Once Ψ_i at θ_0 has been propagated to Ψ at θ, the spin of the particle can be computed as $\vec{S} = \Psi^\dagger\underline{\vec{\sigma}}\Psi$. Alternatively, one can propagate the spinor $\Psi_i = (1,0)^T$ to obtain $\Psi = (a_0 -$

$ia_3, -ia_1 + a_2)^T$ from equation (41). From the real and imaginary parts one then obtains the spin transport quaternion, which makes this method equivalent to the transportation of quaternions in section 2.1.5.

A phase space function $\Psi_{\vec{f}}(\vec{z}, \theta)$ with $|\psi_1|^2 + |\psi_2|^2 = 1$ can describe a spin field if it satisfies the equation of motion

$$\frac{d}{d\theta}\Psi_{\vec{f}}(\vec{z}, \theta) = \partial_\theta \Psi_{\vec{f}}(\vec{z}, \theta) + [\vec{v}(\vec{z}, \theta) \cdot \partial_{\vec{z}}]\Psi(\vec{z}, \theta) = -i\frac{1}{2}[\vec{\Omega}(\vec{z}, \theta) \cdot \underline{\vec{\sigma}}]\Psi(\vec{z}, \theta) \ . \tag{44}$$

In analogy to equation (28), such a spin field is transported by the spin transport quaternion $A(\vec{z}_i, \theta_0, \theta)$ from azimuth θ_0 to θ,

$$\Psi(\vec{z}, \theta) = A(\vec{z}_i, \theta_0; \theta)\Psi(\vec{z}_i, \theta_0) \ \text{ with } \ \vec{z}_i = \vec{M}(\vec{z}, \theta; \theta_0) \ . \tag{45}$$

2.2 Spin Motion in Circular Accelerators

2.2.1 Spin Motion on the Closed Orbit and Imperfection Resonances

Before I analyze spin motion on a general particle trajectory in a circular accelerator, I now take a look at spin motion on the closed orbit. If no field errors, misaligned elements, or energy deviations are present, this orbit is the design trajectory of the accelerator. After a particle has traveled one turn along the closed orbit from azimuth θ_0 to azimuth $\theta_0 + 2\pi$ the spin has rotated around some unit rotation axis $\vec{n}_0(\theta_0)$ by a rotation angle $2\pi\nu_0$. The angle of rotation around $\vec{n}_0$ divided by 2π is called the closed orbit spin tune ν_0 and does not depend on the azimuth θ_0 at which $\vec{n}_0$ is determined. This spin rotation for the closed orbit $\vec{z} = 0$ is described by the spin transport matrix $\underline{R}(0, \theta_0; \theta_0 + 2\pi)$. In the following discussion θ_0 is an arbitrary but fixed azimuth which will no longer be indicated.

In a flat accelerator without field errors and misaligned elements, the closed orbit is in the horizontal plane and passes only through vertical fields. Therefore $\vec{n}_0$ is vertical and $\nu_0 = G\gamma$. When ν_0 is close to an integer, a case which is referred to as an *imperfection resonance*, the rotation matrix is close to the identity and spin directions have hardly changed after one turn. Misalignments create horizontal field components on the design orbit of a flat ring, which produce spin precessions away from the vertical direction. For small misalignments, these rotations around the horizontal might be very small but they can still dominate spin motion when the main fields hardly produce any spin rotation during one turn, i.e. close to integer values of ν_0. Thus the rotation axis $\vec{n}_0$ for spins is vertical away from imperfection resonances but it can be nearly horizontal in their vicinity. At a fixed azimuth θ_0 , the rotation axis $\vec{n}_0$ changes smoothly with ν_0 in between these extremes.

When a particle's energy is accelerated such that ν_0 crosses an integer value, the rotation vector $\vec{n}_0$ can strongly change with energy. When the spin rotation is much faster than this change of the rotation vector, then a spin which is nearly parallel to $\vec{n}_0$ is dragged along with the changing $\vec{n}_0$. The projection of a spin on $\vec{n}_0$ hardly changes during this procedure and can be shown to be an adiabatic invariant [15]. To illustrate this fact, one can imagine that $\vec{n}_0$ changes away from the spin sometimes and towards the spin at other times while the spin rotates around $\vec{n}_0$. Due to this rapid rotation, both cases happen in frequent change and the total effect averages out. This causes the spin to follow the slow change of $\vec{n}_0$.

Since it is inadvisable to let misalignments dominate spin motion, imperfection resonances ought to be avoided. However, since $\nu_0 = G\gamma$ in a flat ring, the closed–orbit spin tune changes during acceleration and the crossing of imperfection resonances is unavoidable. There are the following three possible regimes for resonance crossing:

- If the effects of misalignments are very small, the resonance can be crossed so rapidly that the spins hardly react and the beam's polarization is hardly changed.
- When the effect of misalignments is very strong, the rotation axis $\vec{n}_0$ changes very slowly during acceleration since the precession around the horizontal fields of misaligned elements starts to dominate already far from an imperfection resonance. Then the spin can follow the

slow change of $\vec{n}_0$. But while the average spin direction $< \vec{S} >_N$ changes, the change of the polarization $P = | < \vec{S} >_N |$ is very limited.

- When the effect of misalignments has an intermediate strength, the polarization will be reduced.

The following two strategies can therefore be used to limit the reduction of polarization when imperfection resonances are crossed:

- Careful correction of the closed orbit to limit horizontal field components.

- Increasing the horizontal field components, for example by introducing a solenoid magnet. Devices which are deliberately used to increase the effect of imperfection resonances are referred to as partial snakes [16, 17]. A solenoid magnet has been installed in the AGS and very effectively avoids polarization loss at integer resonances of the closed–orbit spin tune.

Figure 2(left) shows how the spin of a particle on the closed orbit of the DESY III synchrotron would change while it is accelerated from $G\gamma = 7.97$ to $G\gamma = 8.03$ under the influence of a solenoid which rotates the spins by 0.8°. No misalignments are considered. A realistic acceleration rate of 5keV/turn was assumed. Figure 2 (right) shows that the product $s_3 = \vec{S} \cdot \vec{n}_0$ hardly changed during the slow acceleration. This indicates that s_3 is an adiabatic invariant, which was proven in [1]. A small change close to $G\gamma = 8$ recovers after the resonance is crossed. This is not due to the adiabatic invariance but due to the symmetry of spin motion above and below the resonance. The adiabatic following of $\vec{n}_0$ shown in this figure illustrates how a reduction of polarization at imperfection resonances can be avoided.

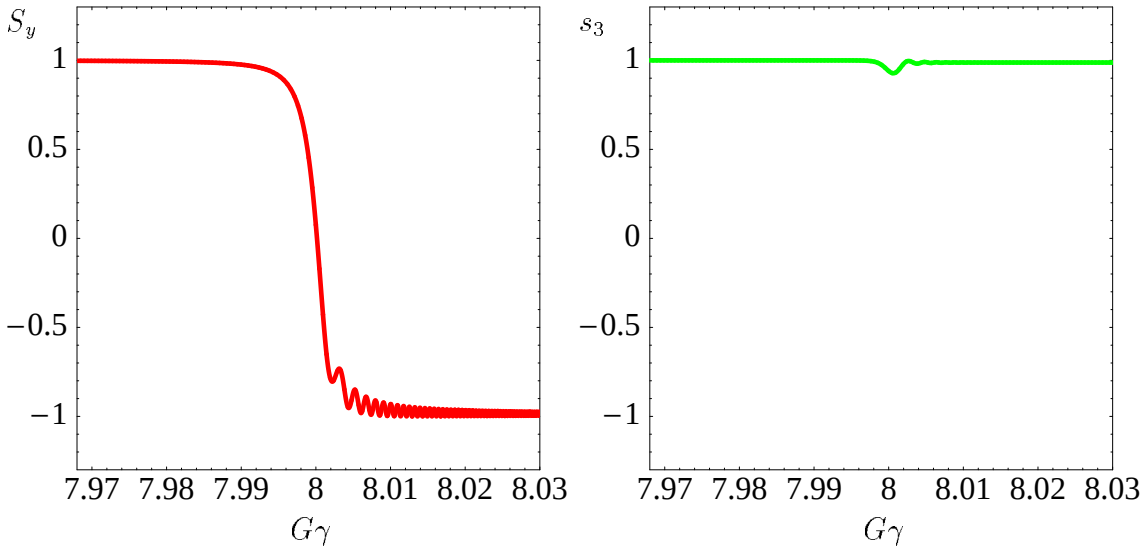

Figure 2: The change of $S_y = \vec{S} \cdot \vec{e}_y$ (left) and the change of $s_3 = \vec{S} \cdot \vec{n}_0$ (right) during the acceleration from $G\gamma = 7.97$ to $G\gamma = 8.03$ for particles on the closed orbit in DESY III in the presence of a 0.8° solenoid partial snake.

For the acceleration process in a simple accelerator model, the change of $\vec{S} \cdot \vec{n}_0$ at a fixed azimuth θ_0 is described by the Froissart–Stora formula as will be discribed in section 2.2.6. This formula allows a quantitative computation of the limited reduction of polarization when either crossing a weak resonance relatively quickly or when crossing a strong resonance relatively slowly.

2.2.2 Spin Motion for Phase Space Trajectories and Intrinsic Resonances

Assuming linearized phase space motion, the particles appear to perform harmonic oscillations around the closed orbit with the frequencies Q_x, Q_y, and Q_τ for horizontal, vertical, and longitudinal motion when viewed at a fixed azimuth θ_0 of the accelerator. These are called the orbital tunes. Some of the fields through which a particle propagates will therefore oscillate with the orbital tunes. Whenever the non–integer part of the spin precession frequency is in resonance with these oscillation frequencies of the particle's coordinates, a severe reduction of polarization can occur. The spin precession frequency of particles moving on the closed orbit is determined by the closed–orbit spin tune ν_0. In general the spin tune is denoted by ν and depends on the amplitude of a particle's oscillations around the closed orbit. Whenever ν is a linear combination of the frequencies of the particle's coordinates, the resulting coherent perturbation can reduce the beam's polarization,

$$\nu = j_0 P_s + j_1 Q_x + j_2 Q_y + j_3 Q_\tau \quad , \quad P_s, j_n, \in \mathbb{N} \ . \tag{46}$$

A super–periodicity P_s of a ring reduces the number of resonances. These resonances are called *intrinsic resonances* of order n for $n = |j_1| + |j_2| + |j_3|$. The depolarizing effect of these resonances has been experimentally verified in many low energy polarized proton accelerators [18]. The first order intrinsic resonances are the dominant reason for a reduction of polarization after solenoids have been introduced to eliminate the effect of imperfection resonances. If the first–order resonances are avoided, however, higher–order resonances become dominant even for decoupled linear phase space motion. This is especially true at high energies and has been shown in detail for HERA-p in [1, 6].

It has been explained in section 2.2.1 that the polarization can be reduced at imperfection resonances due the fact that field imperfections dominate the spin motion whenever the main guide fields produce an integer number of spin rotations, and therefore no apparent spin rotation after one completed turn. The depolarizing effect at intrinsic resonances can be understood in similar terms. For phase space trajectories which deviate little from the closed orbit, the spin motion is dominated by the main guide fields on the closed orbit except close to an intrinsic resonance, where the coherent perturbations described above can dominate over the main guide fields.

To illustrate for example the $\nu_0 = Q_y$ resonance, the spin directions are expressed in terms of a coordinate system which rotates by 2π around $\vec{n}_0$ during one betatron period of vertical motion. In this coordinate system the main guide fields produce a rotation of the spins by $2\pi(\nu_0 - Q_y)$ during one turn. At $\nu_0 = Q_y$ the spin rotation due to the main guide fields vanishes and the remaining rotations are due to extra fields picked up by the oscillating trajectory some distance away from the closed orbit. At intrinsic resonances these spurious effects dominate over the effect of the accelerator's main guide fields. Since the dominant rotation at an intrinsic resonance is produced by the fields along a particle's phase space trajectory, it is different for different particles and the beam will therefore loose polarization under the influence of an intrinsic resonance.

I have described that spins on the closed orbit follow any slow change of $\vec{n}_0$ as long as the system does not remain at a resonance for too long. Therefore, a severe reduction of polarization while accelerating through an imperfection resonances can be avoided by making the acceleration rate slow enough or by making the change of $\vec{n}_0$ slow enough by means of a partial snake as discussed in section 2.2.1.

At intrinsic resonances a reduction of polarization can be avoided by a similar mechanism. If a strong coherent perturbation is slowly switched on and off, an effect similar to adiabatically following $\vec{n}_0$ occurs and polarization is conserved. While an intrinsic resonance is crossed, perturbations influencing particles in the tails of a beam will slowly increase already before the resonance and an adiabatic conservation of polarization can occur. Polarization in the core of the beam will be only weakly influenced when crossing intrinsic resonances, but in intermediate parts of the beam, the polarization is reduced. Such a reduction of polarization can be overcome by slowly exciting the whole beam coherently at a frequency close to the orbital tune which causes the perturbation. All spins then follow the adiabatic change of the polarization direction and the resonance can be

crossed with little loss of polarization. The excitation amplitude is then reduced slowly so that the beam emittance does not change noticeably during the whole process. This mechanism has recently been tested successfully at the AGS [19]. There, an RF dipole has been used to slowly excite all the particle amplitudes coherently. Then the dominant resonances $0 + Q_y$, $12 + Q_y$, $36 - Q_y$, and $36 + Q_y$ were crossed with little loss of polarization. Finally the RF dipole was slowly switched off. No noticeable increase of emittance has been observed. An older technique of avoiding the reduction of polarization at strong intrinsic resonances utilizes pulsed quadrupoles to move the orbital tune within a few microseconds just before a resonance so that the resonance is crossed so quickly that the spin motion is hardly disturbed.

For the case of a single resonance with frequency κ which is crossed by changing the closed–orbit spin tune according to $\nu_0 = \kappa + \alpha\theta$, the Froissart–Stora formula to be introduced in section 2.2.7 shows that polarization can be preserved when an intrinsic resonance is crossed either very quickly or very slowly.

A third method of avoiding loss of polarization at intrinsic resonances uses radial magnetic fields. The closed–orbit spin tune ν_0 is then no longer required to be $G\gamma$, in fact it can be made independent of energy and low order resonances can then be avoided during the acceleration process. It was mentioned below equation (5) that in a fixed transverse magnetic field the deflection angle of high energy particles depends on energy, whereas the spin rotation does not depend on energy. It is therefore possible to devise a fixed field magnetic device which rotates spins by π whenever a high energy particle travels through it at the different energies of an acceleration cycle. Such field arrangements which rotate spins by π while perturbing the orbit only moderately are called Siberian Snakes [20, 21]. Figure 3 illustrates how two Siberian Snakes make the spin tune ν_0 independent of energy and equal to $\frac{1}{2}$ in a flat ring. Starting at the far side of the ring, spins are rotated around the vertical (dashed line) by $\Psi = G\gamma\frac{\pi}{2}$ while the particles travel through one quadrant to the left side of the figure. The light arrow represents a spin which is rotated by Ψ whereas the dark arrow is only rotated by the Siberian Snakes and not by the fields in the arcs. The difference between the light and the dark arrow therefore indicates the rotation due to the fields of the quadrants. A radial Siberian Snake rotates all spins by π around the radial direction before the particles enter the second quadrant. Since the spins have now reversed their vertical orientation, the rotation due to the first quadrant is rewound during the second quadrant. The rotation of the third quadrant is rewound during the fourth, due to the longitudinal Siberian Snake between these quadrants. The rotations of different quadrants cancel for all energies. As indicated by the dark area, all spins have in total rotated by π around the vertical by the time they have returned to the far side of the ring. No net rotation due to the arcs remains and the dark arrow and the light arrow therefore coincide.

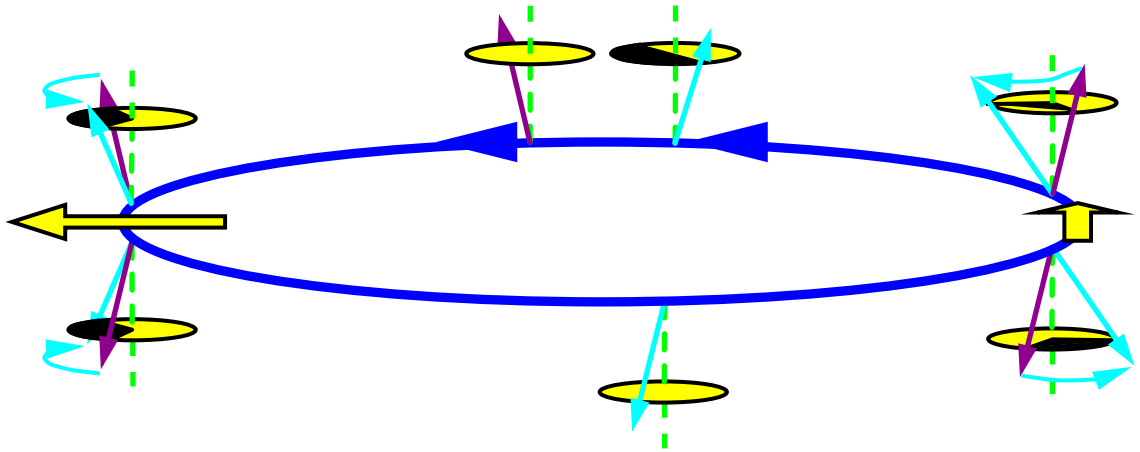

Figure 3: Schematic spin motion in a flat ring with a symmetrically arranged longitudinal and radial snake. The one turn spin motion has $\nu_0 = \frac{1}{2}$ and $\vec{n}_0$ vertical for all energies.

To be more general, one can consider N Siberian Snakes in a ring where a spin rotation angle Ψ_j around the vertical is produced between the $j$the and the $j+1$st Siberian Snake. These angles are in general energy dependent. The rotation axis of a snake is called the snake axis and the angle of this axis to the radial direction is referred to as the snake angle φ_j . The spin transport quaternion of one snake is therefore $i[\cos(\varphi_j)\sigma_1 + \sin(\varphi_j)\sigma_2]$ and the total rotation during one turn is given by

$$A = \prod_{j=1}^{N} i e^{-i\frac{\Psi_j}{2}\sigma_3}[\cos(\varphi_j)\sigma_1 + \sin(\varphi_j)\sigma_2] \quad (47)$$

$$= i^N e^{-i\frac{\Delta\Psi}{2}\sigma_3} \prod_{j=1}^{N}[\cos(\varphi_j)\sigma_1 + \sin(\varphi_j)\sigma_2] \ . \quad (48)$$

Since σ_3 anti–commutes with the other two Pauli matrices, the exponent is given by $\Delta\Psi = \Psi_N - \Psi_{N-1} \pm \ldots \pm \Psi_1$. The total spin rotation is independent of energy when the snake locations are chosen to let $\Delta\Psi = 0$. A pair of snakes produces a rotation around a vertical axis of

$$\begin{aligned} & [\cos(\varphi_1)\sigma_1 + \sin(\varphi_1)\sigma_2] \cdot [\cos(\varphi_2)\sigma_1 + \sin(\varphi_2)\sigma_2] \cdot \\ = \ & \cos(\varphi_1 - \varphi_2) - i\sin(\varphi_1 - \varphi_2)\sigma_3 \ . \end{aligned} \quad (49)$$

An even number of Siberian Snakes therefore produces a vertical rotation vector $\vec{n}_0$. The polarization direction on the closed orbit is then vertical in the bending magnets of the ring and is not deflected in these magnets. For an odd number of Siberian Snakes, $\vec{n}_0$ is in the horizontal plane and rotates by an energy dependent angle in each bending magnet, even though the total rotation of one turn does not depend on energy. The number N is therefore required to be even. In this case the total rotation is given by

$$A = i^N e^{-i(\frac{\Delta\Psi}{2} + \Delta\varphi)\sigma_3} \ , \quad (50)$$

with $\Delta\varphi = \varphi_N - \varphi_{N-1} \pm \ldots - \varphi_1$. For N Siberian Snakes in a ring with otherwise spin rotations only around the vertical, the following three conditions are required:

- $\Delta\Psi = 0$, to make ν_0 independent of energy.
- N is even, to make $\vec{\nu}$ vertical in the arcs of the ring.
- $\Delta\varphi = \frac{\pi}{2}$, to make $\nu_0 = \frac{1}{2}$.

All imperfection resonances and, since the orbital tunes cannot be 1/2, also all first–order intrinsic resonances are avoided by the insertion of such Siberian Snakes, and polarized beam acceleration to very high energy could become possible. Siberian Snakes can only be used at sufficiently high energies since their fields are not changed during acceleration of the beam and they produce orbit distortions which are too big for energies below approximately 8GeV [22]. In high energy storage rings it can be essential to find a sutable choice of snake angles [23, 24].

The orbit deviation in the Siberian Snakes built for RHIC is up to 3cm at injection momentum of about 25GeV/c as shown in figure 4 (left). The orbit motion outside the Siberian Snake, however, is hardly changed by the insertion of this device. One such snake is made of 4 helical dipole magnets of about 2.4m length [25]. Figure 4 (right) depicts the design orbit in a RHIC Siberian Snake in three dimensions. It is obvious why these devices, first suggested in Novosibirsk, received their name.

2.2.3 The Invariant Spin Field

In order to maximize the number of collisions of particles inside the experimental detectors of a storage ring system, one tries to maximize the total number of particles in the bunches and tries to minimize the emittances so that the particle distribution across phase space is narrow and

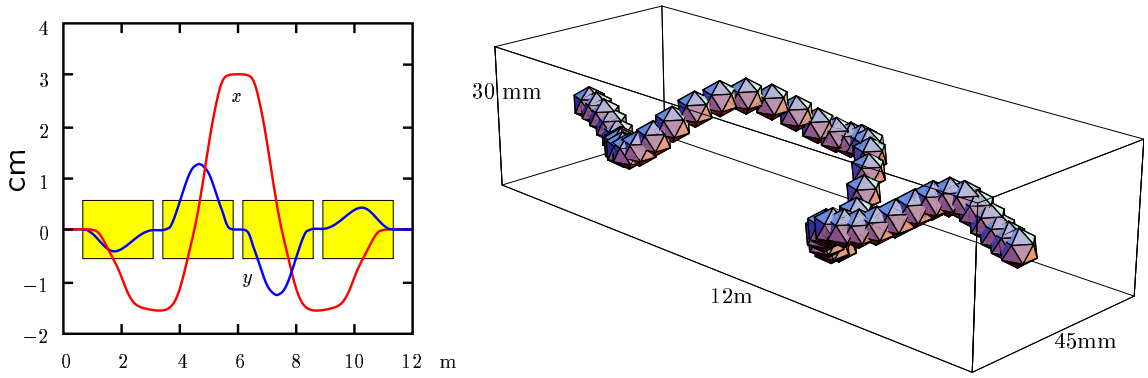

Figure 4: Orbit motion in a helical snake designed for RHIC.

the phase space density is high. If the beam is spin polarized, one additionally requires that the polarization is high and that it does not change much with time.

When all particles of a beam are initially completely polarized parallel to each other, the polarization state of the beam is in general not 2π periodic and the average beam polarization can change from turn to turn. Spin fields are propagated by equation (28). A special spin field $\vec{n}(\vec{z},\theta)$ which is 2π periodic in θ is called an invariant spin field,

$$\vec{n}(\vec{z},\theta) = \underline{R}(\vec{z}_i,\theta_0;\theta)\vec{n}(\vec{z}_i,\theta_0) \ , \quad \vec{n}(\vec{z},\theta+2\pi) = \vec{n}(\vec{z},\theta) \ . \tag{51}$$

If the spin of each particle in a beam is initially polarized parallel to $\vec{n}(\vec{z},\theta_0)$, particles get redistributed in phase space during one turn, but they will stay polarized parallel to the invariant spin field. The beam is then in an equilibrium spin state. Particles change their location in phase space from some initial phase space coordinate $\vec{z}_i$ at azimuth θ_0 to some final coordinate after one turn $\vec{z}_f = \vec{M}(\vec{z}_i)$ according to the one turn map. After one turn, a spin has changed its direction according to the one turn spin transport matrix $\underline{R}(\vec{z}_i) = \underline{R}(\vec{z}_i,\theta_0;\theta_0+2\pi)$, but it is now parallel to the invariant spin field at the particle's new phase space coordinate $\vec{z}_f$, and equation (52) is therefore equivalent to the periodicity condition

$$\vec{n}(\vec{M}(\vec{z})) = \underline{R}(\vec{z})\vec{n}(\vec{z}) \ . \tag{52}$$

The invariant spin field was first introduced by Derbenev and Kondratenko in the theory of radiative electron polarization and is often called the Derbenev–Kondratenko $\vec{n}$–axis. Note that $\vec{n}(\vec{z})$ is usually not an eigenvector of the spin transport matrix $\underline{R}(\vec{z})$ at some phase space point since the spin of a particle has changed after one turn around the ring, but the eigenvector does not change when it is transported by $\underline{R}(\vec{z})$.

The guide fields in storage rings are produced by dipole and quadrupole magnets. The dipole fields constrain the particles to almost circular orbits and the quadrupole fields focus the beam, thus ensuring that the particles do not drift too far away from the central orbit. In these fields, spins precess according to the T–BMT equation (22).

In horizontal dipoles, spins precess only around the vertical field direction. The quadrupoles have vertical and horizontal fields and additionally cause the spins to precess away from the vertical direction. The strength of the spin precession and the precession axis in machine magnets depends on the trajectory and the energy of the particle. Thus in one turn around the ring the effective precession axis can deviate from the vertical and can strongly depend on the initial position of the particle in 6 dimensional phase space of equation (19). From this it is clear that if an invariant spin field $\vec{n}(\vec{z})$ exists, it can vary strongly across the orbital phase space.

Once this field $\vec{n}(\vec{z})$ together with the phase space dependent polarization, its direction, and the phase space density function $\rho(\vec{z})$ of the beam are known, one has a complete specification of the polarization state of a beam of spin 1/2 particles. Maximizing the polarization of the ensemble

implies two conditions: the polarization at each point in phase space should be high and the polarization vector $\vec{n}(\vec{z})$ at each point should be almost parallel to the average polarization vector of the particles.

At very high energy, as for example in the HERA proton ring [26, 27, 28], it can happen that $\vec{n}(\vec{z})$ for particles with realistic phase space amplitudes deviates by tens of degrees from the beam average $< \vec{n} >$ at azimuth θ_0. Thus even if each point in phase space were 100% polarized parallel to $\vec{n}(\vec{z})$, the beam average polarization could be much smaller than 100%. Clearly it is very important to have accurate and efficient methods for calculating $\vec{n}(\vec{z})$ and for ensuring that the spread of $\vec{n}(\vec{z})$ is as small as possible.

2.2.4 The Amplitude Dependent Spin Tune

The closed–orbit spin tune ν_0 has been introduced as the rotation angle of one turn spin motion for particles on the closed orbit. For particles which oscillate around the closed orbit, this rotation angle can depend on the amplitude of their oscillation. For the case that the orbit motion can be described in terms of action and angle variables $\vec{J}$ and $\vec{\Phi}$, as is always the case for stable linear motion, and the tunes Q_j are not in resonance on the invariant torus described by $\vec{J}$, it will now be shown how to define a spin rotation angle which is independent of $\vec{\Phi}$ on that torus. Assuming that an $\vec{n}$–axis exists, one can introduce two unit vectors $\vec{u}_1(\vec{z})$ and $\vec{u}_2(\vec{z})$ to create a right handed dreibein $[\vec{u}_1, \vec{u}_2, \vec{n}]$. The vectors $\vec{u}_1$ and $\vec{u}_2$ are therefore defined up to a rotation around the $\vec{n}$–axis by an arbitrary phase space dependent angle $\phi(\vec{z})$. The spin direction $\vec{S}$ is expressed in terms of this coordinate system by $\vec{S} = s_1\vec{u}_1 + s_2\vec{u}_2 + J_S\vec{n}$. The coefficient J_S is called the spin action and does not change during the particle motion around the ring since the particle transport matrix $\underline{R}(\vec{z})$ is orthogonal and ensures that $J_S = \vec{S} \cdot \vec{n}$ is invariant. The spin motion in this coordinate system is a rotation around the $\vec{n}$–axis by a phase space dependent angle $2\pi\tilde{\nu}(\vec{z})$.

$$\begin{pmatrix} s_{f1} \\ s_{f2} \\ J_S \end{pmatrix} = \begin{pmatrix} \cos(2\pi\tilde{\nu}(\vec{z})) & -\sin(2\pi\tilde{\nu}(\vec{z})) & 0 \\ \sin(2\pi\tilde{\nu}(\vec{z})) & \cos(2\pi\tilde{\nu}(\vec{z})) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_{i1} \\ s_{i2} \\ J_S \end{pmatrix} . \tag{53}$$

The frequency $\tilde{\nu}(\vec{z})$ of the spin rotation is not suited to describe resonant perturbations of spin motion, since it changes from turn to turn with the phase space angles $\vec{\phi}$ of the particle. One can, however, find a spin precession frequency $\nu(\vec{J})$ that only depends on the invariant amplitudes $\vec{J}$ of phase space motion. Since this amplitude dependent spin tune does not change along a particles trajectory, it is suitable to describe the coherent buildup of spin perturbations, and it can thus describe resonance effects.

2.2.5 Maximum Time Average Polarization

If two particles travel along the same trajectory, the angle between their spins does not change. Since $\vec{n}(\vec{z})$ is a spin field at θ_0 and is therefore propagated according to the T–BMT equation (28), a particle which is initially polarized with an angle ϑ with respect to $\vec{n}(\vec{z}_i)$ will have the angle ϑ with respect to $\vec{n}(\vec{z})$ every time it comes back to θ_0. This is due to the fact that the scalar product $J_S(\vec{z}, \vec{S}) = \vec{S} \cdot \vec{n}(\vec{z})$ is an invariant of spin–orbit motion, which can be seen as follows: When $(\vec{z}_i, \vec{S}_i)$ are the initial phase space point and the initial spin of a particle then the final coordinates after one turn around the ring give

$$\begin{aligned} J_S(\vec{z}_f, \vec{S}_f) &= J_S(\vec{M}(\vec{z}_i), \underline{R}(\vec{z}_i)\vec{S}_i) = [\underline{R}(\vec{z}_i)\vec{S}_i] \cdot \vec{n}(\vec{M}(\vec{z}_i)) \\ &= [\underline{R}(\vec{z}_i)\vec{S}_i] \cdot [\underline{R}(\vec{z}_i)\vec{n}(\vec{z}_i)] = \vec{S}_i \cdot \vec{n}(\vec{z}_i) = J_S(\vec{z}_i, \vec{S}_i) . \end{aligned} \tag{54}$$

Whenever the particle comes back to θ_0 with a phase space coordinate which is close to $\vec{z}_i$, the spin will again have the angle ϑ with respect to $\vec{n}(\vec{z}_i)$, assuming $\vec{n}(\vec{z})$ is sufficiently continuous. Since the components perpendicular to the $\vec{n}$–axis average to zero after many turns, the time averaged polarization at $\vec{z}_i$ will be parallel to $\vec{n}(\vec{z}_i)$, and it can only have the magnitude 1 if the spin was initially parallel to the invariant spin field. However, even if all particles are initially polarized

parallel to $\vec{n}(\vec{z})$, the beam polarization is not 1 but $|<\vec{n}>|$ where $<\ldots>$ denotes the average over the beam. The maximum time average beam polarization that can be stored in an accelerator at a given fixed energy is therefore $P_{lim} = |<\vec{n}>|$.

2.2.6 The Single Resonance Model (SRM)

In the previous sections, $\vec{n}$, $\vec{u}_1$, $\vec{u}_2$, and ν have been introduced and the adiabatic invariance of J_S has been described. Now the introduced quantities will be computed for an analytically solvable model and the adiabatic invariance will be illustrated by examining changes of a parameter of this model.

For simplicity, a coordinate system is introduced which has the one turn rotation axis $\vec{n}_0$ as one of its coordinate vectors and in which the spin motion on the closed orbit is as simple as possible.

The precession vector $\vec{\Omega}(\vec{z},\theta)$ on the closed orbit $\vec{z}=0$ is denoted by $\vec{\Omega}_0(\theta)$. The rotation axis $\vec{n}_0(\theta_0)$ of the one turn spin transport matrix is sometimes called the spin closed orbit [13]; it satisfies

$$\frac{d}{d\theta}\vec{n}_0(\theta) = \vec{\Omega}_0(\theta)\times\vec{n}_0(\theta)\ , \quad \vec{n}_0(\theta) = \vec{n}_0(\theta+2\pi)\ . \tag{55}$$

Two unit vectors $\vec{m}_0(\theta)$ and $\vec{l}_0(\theta)$ are now chosen which initially make up a right handed orthogonal dreibein $[\vec{m}_0(\theta_0),\vec{l}_0(\theta_0),\vec{n}_0(\theta_0)]$ and propagate around the ring according to the T–BMT equation on the closed orbit,

$$\frac{d}{d\theta}\vec{m}_0 = \vec{\Omega}_0(\theta)\times\vec{m}_0\ , \quad \frac{d}{d\theta}\vec{l}_0 = \vec{\Omega}_0(\theta)\times\vec{l}_0\ . \tag{56}$$

The three unit vectors will always constitute a right handed orthogonal dreibein, since all three get rotated by the same precession equation. Whereas $\vec{n}_0$ is periodic around the ring, the vectors $\vec{m}_0$ and $\vec{l}_0$ are rotated around $\vec{n}_0$ by the angle $2\pi\nu_0$ after one turn and the dreibein is therefore in general not 2π periodic in θ. Now a 2π periodic dreibein is defined by rotating $\vec{m}_0$ and $\vec{l}_0$ back uniformly by $2\pi\nu_0$ during one turn [30, 31],

$$\vec{m}+i\vec{l} = e^{i\nu_0\theta}(\vec{m}_0+i\vec{l}_0)\ , \quad \frac{d}{d\theta}(\vec{m}+i\vec{l}) = (\vec{\Omega}_0-\nu_0\vec{n}_0)\times(\vec{m}+i\vec{l})\ . \tag{57}$$

In this coordinate system, a spin can be written as

$$\vec{S}(\theta) = s_1(\theta)\vec{m}(\theta)+s_2(\theta)\vec{l}(\theta)+s_3(\theta)\vec{n}_0(\theta)\ , \quad s_1^2+s_2^2+s_3^2 = 1\ . \tag{58}$$

The equation of spin motion

$$\vec{\Omega}_0\times\vec{S} = \frac{d}{d\theta}\vec{S} = \vec{m}\frac{d}{d\theta}s_1+\vec{l}\frac{d}{d\theta}s_2+\vec{n}_0\frac{d}{d\theta}s_3+(\vec{\Omega}_0-\nu_0\vec{n}_0)\times\vec{S} \tag{59}$$

can be decomposed into its components parallel to $\vec{m}$, $\vec{l}$, and $\vec{n}_0$, which leads to

$$\frac{d}{d\theta}(s_1+is_2) = i\nu_0(s_1+is_2)\ , \quad \frac{d}{d\theta}s_3 = 0 \tag{60}$$

and describes a uniform rotation around $\vec{n}_0$, which keeps s_3 invariant. In fact, s_3 is an adiabatic invariant since it does not change much even when parameters of the spin motion, like the particles' energy, are slowly changed.

The spin precession vector for particles which oscillate around the closed orbit can be decomposed in the closed orbit contribution $\vec{\Omega}_0$ and a part $\vec{\omega}$ due to the oscillation amplitude, $\vec{\Omega}(\vec{z},\theta) = \vec{\Omega}_0(\theta)+\vec{\omega}(\vec{z},\theta)$. The one turn rotation axis $\vec{n}_0$ precesses around $\vec{\Omega}_0$ and the 2π periodic dreibein $[\vec{m},\vec{l},\vec{n}_0]$ precesses according to equation (57) around $\vec{\Omega}_0-\nu_0\vec{n}_0$.

When the motion in phase space can be transformed to action–angle variables, the spin precession vector $\vec{\omega}(\vec{J},\vec{\Phi},\theta)$ for particles which oscillate around the closed orbit is a 2π periodic function of $\vec{\Phi}$ and θ . With the possibly amplitude dependent orbit tunes $\vec{Q} = \frac{d}{d\theta}\vec{\Phi}$, the Fourier spectrum

of ω has frequencies $j_0 + \vec{j} \cdot \vec{Q}$ with integers j_n . The integer contributions j_0 are due to the 2π periodicity of $\vec{\omega}$ with θ and the contributions of integer multiples of the obit tunes Q_k are due to the periodicity of $\vec{\omega}$ with Φ_k . When one of the Fourier frequencies is nearly in resonance with ν_0, it can be a good approximation to drop all other Fourier components. This is referred to as the single resonance approximation. However, this can only be good approximation when the influence of individual resonances is well separated. That model corresponds to the rotating field approximation often used to discuss spin resonance in solid state physics [32]. For a conventional flat ring, the first–order resonance due to vertical motion dominates and therefore most often the resonance $\kappa = j_0 \pm Q_y$ is considered.

The amplitude of the one remaining Fourier contribution is called the resonance strength ϵ_κ. For first–order resonances, where $\sum_{n=1}^{3} |j_n| = 1$, ϵ_κ is computed in section 3.2. A method for computing higher–order resonance strength, where $\sum_n = 1^3 |j_n| > 1$ can be found in [1].

The analytically solvable model which is now considered consists of $\vec{\Omega}_0 = \nu_0 \vec{n}_0$ and of an $\vec{\omega}$ which only has one Fourier contribution, $\vec{\omega} = \epsilon_\kappa (\vec{m} \cos\Phi + \vec{l} \sin\Phi)$, with $\Phi = j_0\theta + \vec{j} \cdot \vec{\Phi} + \Phi_0$. Since $\frac{d}{d\theta}\vec{\Phi} = \vec{Q}$, the frequency is $\kappa = j_0 + \vec{j} \cdot \vec{Q}$. When the coordinates in the $[\vec{m}, \vec{l}, \vec{n}_0]$ system are arranged in column vectors [29, 33], one obtains

$$\frac{d}{d\theta}\Phi = \kappa \ , \quad \frac{d}{d\theta}\vec{s} = \vec{\Omega}(\Phi) \times \vec{s} \ , \quad \vec{\Omega} = \begin{pmatrix} \epsilon_\kappa \cos\Phi \\ \epsilon_\kappa \sin\Phi \\ \nu_0 \end{pmatrix} \ . \tag{61}$$

Initial coordinates $\vec{z}_i$ are taken into final coordinates $\vec{z}_f$ by $\vec{\Phi}_f = \vec{\Phi}_i + 2\pi\vec{Q}$ and therefore $\Phi_f = \Phi_i + 2\pi\kappa$. Now the orthogonal matrix $\underline{T}(\vec{e}, \varphi)$ is introduced to describe a rotation around a unit vector $\vec{e}$ by an angle φ. Transforming the spin components of $\vec{s}$ into a rotating frame by $\vec{s}_R = \underline{T}(\vec{e}_1, -\Phi) \cdot \vec{s}$, one obtains the simplified equation of spin motion

$$\frac{d}{d\theta}\vec{s}_R = \vec{\Omega}_R \times \vec{s}_R \ , \quad \vec{\Omega}_R = \begin{pmatrix} \epsilon_\kappa \\ 0 \\ \delta \end{pmatrix} \ , \quad \delta = \nu_0 - \kappa \ . \tag{62}$$

If a spin field is oriented parallel to $\vec{\Omega}_R$ in this frame, it does not change from turn to turn. Therefore $\vec{n}_R = \vec{\Omega}_R / |\vec{\Omega}_R|$ is an $\vec{n}$–axis. In the original frame, this $\vec{n}$–axis is

$$\vec{n}(\Phi) = \mathrm{sig}(\delta) \frac{1}{\Lambda} \begin{pmatrix} \epsilon_\kappa \cos\Phi \\ \epsilon_\kappa \sin\Phi \\ \delta \end{pmatrix} \ , \quad \Lambda = \sqrt{\delta^2 + \epsilon_\kappa^2} \ , \tag{63}$$

where the 'sign factor' $\mathrm{sig}(\delta)$ has been chosen so that on the closed orbit ($\epsilon_\kappa = 0$) the $\vec{n}$–axis $\vec{n}(\Phi)$ coincides with $\vec{n}_0 = \vec{e}_3$. As with any function of phase space, this $\vec{n}$–axis is a 2π periodic function of the angle variables $\vec{\Phi}$ and of θ . As required, $\vec{n}$ is a solution of the T–BMT equation (61), $\frac{d}{d\theta}\vec{n} = \mathrm{sig}(\delta) Q (\vec{e}_2 \cos\Phi - \vec{e}_1 \sin\Phi) = \vec{\Omega} \times \vec{n}$.

This analytically solvable model can also be used to illustrate the construction of a phase independent but amplitude dependent spin tune $\nu(\vec{J})$ [1]. It will be seen that this spin rotation angle, and no other angle which might be alternatively proposed [14], determines the location of resonances. Having got an $\vec{n}$–axis, one can transform the components of $\vec{s}$ into a coordinate system $[\vec{n}, \vec{\tilde{u}}_1, \vec{\tilde{u}}_2]$. With the simple choice

$$\vec{\tilde{u}}_2(\Phi) = \frac{\vec{e}_3 \times \vec{n}}{|\vec{e}_3 \times \vec{n}|} = \mathrm{sig}(\delta) \begin{pmatrix} -\sin\Phi \\ \cos\Phi \\ 0 \end{pmatrix} \ , \quad \vec{\tilde{u}}_1(\Phi) = \frac{1}{\Lambda} \begin{pmatrix} \delta \cos\Phi \\ \delta \sin\Phi \\ -\epsilon_\kappa \end{pmatrix} \ , \tag{64}$$

$\vec{\tilde{u}}_1$ is equal to $\vec{\tilde{u}}_2 \times \vec{n}$ and the basis vectors are clearly 2π periodic in $\vec{\Phi}$ and in θ as required. Since the basis vectors $\vec{\tilde{u}}_1$ and $\vec{\tilde{u}}_2$ build an orthogonal dreibein with $\vec{n}$ for all θ, and since $\vec{n}$ precesses around $\vec{\Omega}$, one obtains $\frac{d}{d\theta}\vec{\tilde{u}}_2 = (\vec{\Omega} - \tilde{\nu}\vec{n}) \times \vec{\tilde{u}}_2$ for some function $\tilde{\nu}$ of phase space,

$$\tilde{\nu} = (\frac{d}{d\theta}\vec{\tilde{u}}_2 - \vec{\Omega} \times \vec{\tilde{u}}_2) \cdot \vec{\tilde{u}}_1 = \mathrm{sig}(\delta) [\begin{pmatrix} -\kappa \cos\Phi + \nu_0 \cos\Phi \\ -\kappa \sin\Phi + \nu_0 \sin\Phi \\ -\epsilon_\kappa \end{pmatrix}] \cdot \vec{\tilde{u}}_1 = \mathrm{sig}(\delta) \Lambda \ . \tag{65}$$

On the design orbit this rotation angle is $\delta = \nu_0 - \kappa$. In order to let the amplitude dependent spin tune reduces to ν_0 on the design orbit, we can choose a coordinate system $[\vec{n}, \vec{u}_1, \vec{u}_2]$ that rotates around $\vec{n}$ by κ during one turn with respect to the original coordinate system $[\vec{n}, \vec{\tilde{u}}_1, \vec{\tilde{u}}_2]$. This is achieved by rotating the basis vectors with ϕ,

$$\vec{u}_1 = \vec{\tilde{u}}_1 \cos\Phi - \vec{\tilde{u}}_2 \sin\Phi \ , \quad \vec{u}_2 = \vec{\tilde{u}}_2 \cos\Phi + \vec{\tilde{u}}_1 \sin\Phi \ . \tag{66}$$

The spin rotation angle then becomes

$$\nu = \mathrm{sig}(\delta)\Lambda + \kappa \ . \tag{67}$$

In the SRM, $\epsilon_\kappa = |\vec{\omega}(\vec{z})|$, and therefore $\tilde{\nu}$ depends on the orbital amplitude and $\nu(\vec{J})$ is the amplitude dependent spin tune.

On the closed orbit, the coordinate system now reduces to

$$\vec{n} \to \vec{n}_0 \ , \quad \vec{u}_1 \to \mathrm{sig}(\delta)\vec{m} \ , \quad \vec{u}_2 \to \mathrm{sig}(\delta)\vec{l} \ , \quad \nu \to \nu_0 \ . \tag{68}$$

This model leads to the average polarization

$$P_{lim} = | < \vec{n}(\vec{z}) > | = \frac{|\delta|}{\sqrt{\delta^2 + \epsilon_\kappa^2}} = \sqrt{1 - \left(\frac{\epsilon_\kappa}{\Delta}\right)^2} \ , \ \Delta = \nu - \kappa \ , \ \delta = \nu_0 - \kappa \tag{69}$$

which is plotted in figure 5 (top). The distance of the amplitude dependent spin tune ν from resonance has here been denoted by Δ, which is equivalent with $\tilde{\nu} = \mathrm{sig}(\delta)\Lambda$. In the bottom figure, the spin tune ν of equation (67) jumps by $2\epsilon_\kappa$ at the resonance where $\nu_0 = \kappa$. This jump of the spin tune could in general be transformed away since the sign of the spin tune is not uniquely determined. This however requires a change of the sign of $\vec{n}$. Here the sign of $\vec{n}$ in equation (63) has been fixed by choosing $\vec{n}_0 \cdot \vec{n} > 0$; and the tune jump at resonance can therefore not be transformed away.

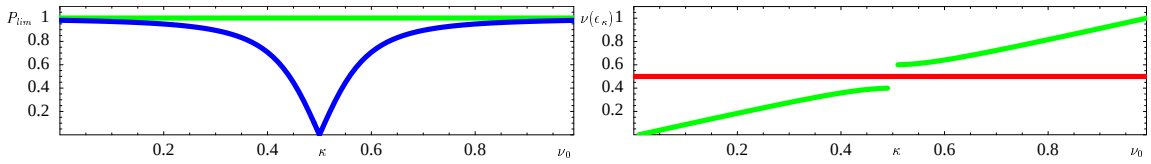

Figure 5: P_{lim} and the amplitude dependent spin tune $\nu(\epsilon_\kappa)$ for the SRM in the vicinity of $\nu_0 = \kappa$, for $\kappa = 0.5$ and $\epsilon_\kappa = 0.1$.

2.2.7 The Froissart–Stora Formula

The adiabatic spin invariant was established for general systems in section [1]. For the analytically solvable SRM the change of this adiabatic invariant can be computed. When the closed–orbit spin tune ν_0 changes during the acceleration process, intrinsic resonances and imperfection resonances have to be crossed. While the spin is under the strong influence of an approximately resonant Fourier contribution of ω, a reduction of polarization can occur which does not recover after the energy has increased and the resonance is crossed.

The reduction of polarization during resonance crossing is traditionally described in the framework of the SRM by the Froissart–Stora formula. To describe resonance crossing, a changing closed–orbit spin tune ν_0 has to be inserted in the equation of motion (61). For various functions $\nu_0(\theta)$, different approaches are possible [34, 35, 13]. If the closed–orbit spin tune changes like $\nu_0 = \kappa + \alpha\theta$, the corresponding spinor equation of motion (39) can be solved in terms of confluent hypergeometric functions. The equations for arbitrary initial conditions are quite complicated

but when at $\theta \to -\infty$ a vertical spin $s_3(-\infty) = 1$ is chosen as initial condition then the vertical component at $\theta \to +\infty$ is given by the Froissart–Stora formula

$$s_3(\infty) = 2e^{-\pi \frac{\epsilon_\kappa^2}{2\alpha}} - 1 \ . \tag{70}$$

In the case of a strong perturbation ϵ_κ, or when the acceleration is very slow, spins follow the change of $\vec{n}(\Phi)$. The $\vec{n}$–axis in equation (63) has a discontinuity from $\vec{n}_- = -\epsilon_\kappa(\vec{e}_1 \cos\Phi + \vec{e}_2 \sin\Phi)$ just below resonance to $\vec{n}_+ = -\vec{n}_-$ just above resonance. Spins do not follow this instantaneous change of sign, but they then follow $-\vec{n}$ adiabatically after the resonance has been crossed. Therefore $s_3(\infty)$ is close to -1 for a slow change of ν_0 . When the perturbation is weak or crossed very quickly, then spin motion is hardly affected and $s_3(\infty)$ is close to 1 in equation (70). In intermediate cases, the polarization is reduced.

3 First–Order Spin Motion

3.1 Linearized Spin–Orbit Motion

At azimuth θ, a spin can be described by a complex coordinate α with

$$\vec{S} = \mathrm{Re}\{\alpha\}\vec{m}(\theta) + \mathrm{Im}\{\alpha\}\vec{l}(\theta) + \sqrt{1-|\alpha|^2}\vec{n}_0(\theta) \ , \tag{71}$$

where the right handed orthonormal dreibein $[\vec{m}, \vec{l}, \vec{n}_0]$ is used which was introduced in section 2.2.6 [30, 31]. The coordinate vectors $\vec{m}$ and $\vec{l}$ satisfy the equation of motion

$$\frac{d}{d\theta}(\vec{m} + i\vec{l}) = (\vec{\Omega}_0 - \nu_0\vec{n}_0) \times (\vec{m} + i\vec{l}) \ . \tag{72}$$

The spin of a particle which travels on the closed orbit precesses around $\vec{\Omega}_0$ and has rotated ν_0 times around $\vec{n}_0$ after one turn. According to equation (72), $\vec{m}$ and $\vec{l}$ also precess around $\vec{\Omega}_0$, but in addition a precession around $\vec{n}_0$ is subtracted, leaving no net rotation after one turn. Therefore, the dreibein $[\vec{m}, \vec{l}, \vec{n}_0]$ is 2π periodic in θ.

When the spin coordinate α and the phase space coordinates $\vec{z}$ are small so that the equation of spin–orbit motion can be linearized, then one approximates an initial spin of a particle at azimuth θ_0 by $\vec{S}_i \approx \mathrm{Re}\{\alpha_i\}\vec{m}(\theta_0) + \mathrm{Im}\{\alpha_i\}\vec{l}(\theta_0) + \vec{n}_0(\theta_0)$ and after the particle has traveled to azimuth θ, one has $\vec{S} = \mathrm{Re}\{\alpha\}\vec{m}(\theta) + \mathrm{Im}\{\alpha\}\vec{l}(\theta) + \vec{n}_0(\theta)$ where α is determined by the 7×7 spin–orbit transport matrix,

$$\begin{pmatrix} \vec{z} \\ \alpha \end{pmatrix} = \underline{M}_{77}(\theta_0;\theta) \begin{pmatrix} \vec{z}_i \\ \alpha_i \end{pmatrix} = \begin{pmatrix} \underline{M}(\theta_0;\theta) & \vec{0} \\ \vec{G}^T(\theta_0;\theta) & e^{i\nu_0(\theta-\theta_0)} \end{pmatrix} \begin{pmatrix} \vec{z}_i \\ \alpha_i \end{pmatrix} \ , \tag{73}$$

where $\underline{M}(\theta_0;\theta)$ is the 6×6 dimensional transport matrix for the phase space variables. For a particle on the closed orbit, the exponential describes the rotation of the spin component α around $\vec{n}_0$ with respect to $\vec{m}$ and $\vec{l}$. This rotation appears in equation (73) since spins precess around $\vec{\Omega}_0$ for $\vec{z} = 0$, while the coordinate vectors $\vec{m}$ and $\vec{l}$ rotate around $\vec{\Omega}_0 - \nu_0\vec{n}_0$. The complex row vector $\vec{G}^T(\theta_0;\theta)$ describes additional spin motion with respect to $\vec{m}$ and $\vec{l}$ due to off closed–orbit fields. The 6 dimensional zero vector $\vec{0}$ shows that the effect of Stern Gerlach forces on the orbit motion is not considered.

The linearized spin–orbit transport through two successive optical elements is described by the product of their 7×7 matrices. These matrices were derived long ago [30, 36] for all standard optical elements and were initially used for the description of polarized electron beams.

Alternatively, the spin transport can be described by a spin transport quaternion as discussed in section 2.1.5. When linearizing with respect to phase space variables, indicated by $=_1$, a spin transport quaternion $\vec{C} =_1 \vec{C}^0 + \vec{C}^1(\vec{z})$ is separated into the quaternion for closed orbit motion $\vec{C}^0$ and a contribution $\vec{C}^1$ which is linear in the phase space variables. The spin–orbit transport through two successive optical elements is described by the action of first the quaternion $\vec{A}$ associated with

the first element and then the quaternion $\vec{B}$ of the second element. The quaternion $\vec{C}$ describing the combined rotation is computed using the orthogonal 4×4 matrix $\underline{\tilde{A}}$ as described in equation (34),

$$\vec{C}^0 = \underline{\tilde{A}}^0 \vec{B}^0 \ , \quad \vec{C}^1 = \underline{\tilde{A}}^0 \vec{B}^1 + \underline{\tilde{A}}^1 \vec{B}^0 \ . \tag{74}$$

The spin transfer quaternion of an optical element does not depend on the basis vectors $[\vec{m}, \vec{l}, \vec{n}_0]$ and is therefore the same for two identical optical elements which are at different locations of the ring. The 7×7 matrix of individual optical elements does not have this advantage.

The spin transport quaternion can be written as the concatenation of first the closed orbit spin transport described by $\vec{C}^0$ and then a purely phase space dependent spin transport which does not change the spin of particles on the closed orbit. With the quaternion $\vec{e}_1 = (1, 0, 0, 0)^T$ describing the identity transformation and with a quaternion $\vec{D}^1$ which vanishes for particles on the closed orbit, the purely phase space dependent spin transport is described by $\vec{e}_1 + \vec{D}^1$,

$$\vec{C} = \underline{\tilde{C}}^0 (\vec{e}_1 + \vec{D}^1) \ , \quad \vec{D}^1 = \underline{\tilde{C}}^{0T} \vec{C}^1 \ . \tag{75}$$

Advantage has here been taken of the fact that the 4×4 dimensional matrix $\underline{\tilde{C}}^0$ is orthogonal, as has been pointed out after equation (34), and thus inverted by transposition. The 3×3 spin rotation matrix on the closed orbit is written as $\underline{R}^0$ and the rotation matrix corresponding to the concatenated quaternion in equation (75) is the product $\underline{R}^D \underline{R}^0$. Equation (30) relates the quaternion $\vec{e}_1 + \vec{D}^1$ with $\vec{D}^1 = (d_0^1, \vec{d}^{\,1})^T$ with the rotation matrix $\underline{R}^D$, which to first order, becomes

$$R_{ij}^D = [(1 + d_0^1)^2 - (\vec{d}^{\,1})^2]\delta_{ij} + 2d_i^1 d_j^1 - 2(1 + d_0^1)\epsilon_{ijk} d_k^1 =_1 (1 + 2d_0^1)\delta_{ij} - 2\epsilon_{ijk} d_k^1 \ . \tag{76}$$

By the total spin rotation $\underline{R}^D \underline{R}^0$, the initial spin $\vec{S}_i = \vec{n}_0(\theta_0)$ is first transported to $\vec{n}_0(\theta) = \underline{R}^0 \vec{n}_0(\theta_0)$ and then to

$$\vec{S}_f = (1 + 2d_0^1)\vec{n}_0(\theta) + 2\vec{d}^{\,1} \times \vec{n}_0(\theta) \ . \tag{77}$$

When a spin with $\vec{S}_i = \vec{n}_0$ is transported by the 7×7 spin–orbit transport matrix, then $\alpha_f = \vec{S}_f \cdot [\vec{m}(\theta) + i\vec{l}(\theta)]$ is given by $\vec{G} \cdot \vec{z}$ which now equates to

$$\vec{G} \cdot \vec{z} = 2[\vec{d}^{\,1} \times \vec{n}_0(\theta)] \cdot [\vec{m}(\theta) + i\vec{l}(\theta)] = -i2\vec{d}^{\,1} \cdot [\vec{m}(\theta) + i\vec{l}(\theta)] \ . \tag{78}$$

This illustrates how the spin–orbit transport matrix $\underline{M}_{77}$ can easily be computed when its spin transfer quaternion is known.

3.1.1 The Invariant Spin Field for Linearized Spin–Orbit Motion

Although it is difficult to compute $\vec{n}$ in general, an approximation for $\vec{n}$ at azimuth θ_0 can easily be obtained [31, 29, 38] for linearized spin–orbit motion. Its components perpendicular to $\vec{n}_0(\theta_0)$ are written as a complex function $n_\alpha(\vec{z})$ and use a 7 dimensional vector $\vec{n}_1$ to obtain the first–order expansion of $\vec{n}(\vec{z})$. Using the one turn matrix $\underline{M}_{7\times7} = M_{7\times7}(\theta_0; \theta_0 + 2\pi)$, the linearized periodicity condition for the invariant spin field is

$$\vec{n}_1(\vec{z}) = \begin{pmatrix} \vec{z} \\ \alpha_n(\vec{z}) \end{pmatrix} \ , \quad \vec{n}_1(\underline{M}\vec{z}) = \underline{M}_{77} \vec{n}_1(\vec{z}) \ . \tag{79}$$

This equation can be solved for $\vec{n}_1$ after the matrices are diagonalized. Let $\underline{A}^{-1}$ be the column matrix of eigenvectors $\vec{v}_k^{\pm}$ of the one turn matrix $\underline{M}$. The eigenvalues are $e^{\pm i2\pi Q_k}$ with the orbital tunes Q_k. The matrix $\underline{\Lambda} = \underline{A}\ \underline{M}\ \underline{A}^{-1}$ is the diagonal matrix of these eigenvalues. Now the 7×6 dimensional matrix $\underline{T}$ is needed which is the column matrix of the first 6 eigenvectors of $\underline{M}_{77}$ and has the form

$$\underline{T} = \begin{pmatrix} \underline{A}^{-1} \\ \vec{B}^T \end{pmatrix} \ , \quad \underline{T}\ \underline{\Lambda} = \underline{M}_{77} \underline{T} \ , \tag{80}$$

where the 7th components of the eigenvectors form a vector $\vec{B}$. If a linear function $\vec{n}_1(\vec{z}) = \underline{K}\vec{z}$ of the phase space coordinates can be found which satisfies the periodicity condition (79), then an

invariant spin field has been determined. Since the upper 6 components of $\vec{n}_1(\vec{z})$ are $\vec{z}$, the upper 6 rows of $\underline{K}$ form the identity matrix $\underline{1}_6$. Inserting $\vec{n}_1 = \underline{K}\vec{z}$ into equation (79) and multiplying the resulting condition $\underline{K}\ \underline{M} = \underline{M}_{77}\underline{K}$ by $\underline{A}^{-1}$ from the right leads to $\underline{K}\ \underline{A}^{-1}\underline{\Lambda} = \underline{M}_{77}\underline{K}\ \underline{A}^{-1}$. Therefore the columns of $\underline{K}\ \underline{A}^{-1}$ are eigenvectors of $\underline{M}_{7\times 7}$ and are therefore proportional to the columns of $\underline{T}$. The upper 6 rows $\underline{1}_6\underline{A}^{-1}$ agree with those of $\underline{T}$; this requires the 6 proportionality constants to be 1. Therefore $\underline{K}\ \underline{A}^{-1} = \underline{T}$ and I conclude that there exists a unique linear invariant spin field given by

$$\vec{n}_1(\vec{z}) = \underline{T}\ \underline{A}\vec{z}\ , \quad \alpha_n = \vec{B}\cdot(\underline{A}\vec{z})\ . \tag{81}$$

Now the steps which lead to the amplitude dependent spin tune are performed for linearized spin–orbit motion. Together with $\vec{n}(\vec{z}) =_1 \mathrm{Re}\{\alpha_n\}\vec{m} + \mathrm{Im}\{\alpha_n\}\vec{l} + \vec{n}_0$, the following two basis vectors build an orthonormal dreibein in linear approximation at θ_0:

$$\vec{u}_1(\vec{z}) =_1 \vec{m} - \mathrm{Re}\{\alpha_n\}\vec{n}_0\ , \quad \vec{u}_2(\vec{z}) =_1 \vec{l} - \mathrm{Im}\{\alpha_n\}\vec{n}_0\ . \tag{82}$$

A spin $\vec{S}_i =_1 \mathrm{Re}\{\alpha_i\}\vec{m} + \mathrm{Im}\{\alpha_i\}\vec{l} + \vec{n}_0$ is transported to $\vec{S}_f =_1 \mathrm{Re}\{G + e^{i2\pi\nu_0}\alpha_i\}\vec{m} + \mathrm{Im}\{G + e^{i2\pi\nu_0}\alpha_i\}\vec{l} + \vec{n}_0$ after one turn, where

$$\vec{u}_1(\underline{M}\vec{z}) =_1 \vec{m} - \mathrm{Re}\{G + e^{i2\pi\nu_0}\alpha_n\}\vec{n}_0\ , \quad \vec{u}_2(\underline{M}\vec{z}) =_1 \vec{l} - \mathrm{Im}\{G + e^{i2\pi\nu_0}\alpha_n\}\vec{n}_0\ . \tag{83}$$

At the initial phase space point this leads to the projections $\vec{S}_i\cdot(\vec{u}_1(\vec{z}_i) + i\vec{u}_2(\vec{z}_i)) =_1 \alpha_i - \alpha_n$ and after one turn to $\vec{S}_f\cdot(\vec{u}_1(\vec{z}_f) + i\vec{u}_2(\vec{z}_f)) =_1 e^{i2\pi\nu_0}(\alpha_i - \alpha_n)$. The amplitude dependent spin tune ν in linearized spin–orbit motion is therefore simply given by ν_0 .

The eigenvector condition

$$\underline{M}_{77}\begin{pmatrix} \vec{v}_k^{\pm} \\ B_k^{\pm} \end{pmatrix} = e^{\pm i2\pi Q_k}\begin{pmatrix} \vec{v}_k^{\pm} \\ B_k^{\pm} \end{pmatrix} \tag{84}$$

leads to $\vec{G}\cdot\vec{v}_k^{\pm} + e^{i2\pi\nu_0}B_k^{\pm} = e^{\pm i2\pi Q_k}B_k^{\pm}$. Therefore α_n diverges at first–order intrinsic resonances where $\nu_0 = j_0 \pm Q_k$ due to

$$B_k^{\pm} = \vec{G}\cdot\vec{v}_k^{\pm}/(e^{\pm i2\pi Q_k} - e^{i2\pi\nu_0})\ . \tag{85}$$

In the normal form space belonging to the diagonal matrix $\underline{\Lambda}$, the coordinates are given by the actions J_j and the angle variables Φ_j with

$$A\vec{z} = (\sqrt{J_1}e^{i\Phi_1}, \sqrt{J_1}e^{-i\Phi_1}, \sqrt{J_2}e^{i\Phi_2}, \sqrt{J_2}e^{-i\Phi_2}, \sqrt{J_3}e^{i\Phi_3}, \sqrt{J_3}e^{-i\Phi_3})^T\ . \tag{86}$$

The average over all angle variables on an invariant torus is described by $< \ldots >_{\vec{\Phi}}$. It leads to the average opening angle of

$$< \vartheta(\vec{n},\vec{n}_0) >_{\vec{\Phi}} \approx \mathrm{atan}(\sqrt{< |\alpha_n|^2 >_{\vec{\Phi}}}) = \mathrm{atan}\left(\sqrt{\sum_{k=1}^{3}(|B_k^+|^2 + |B_k^-|^2)J_k}\right)\ , \tag{87}$$

where the $B_k^{\pm}$ are the 7th components of the eigenvectors in equation (80). The maximum time average polarization is approximately

$$P_{lim} =< \cos(\vartheta(\vec{n},\vec{n}_0)) >_{\vec{\Phi}} \approx [1 + \sum_{k=1}^{3}(|B_k^+|^2 + |B_k^-|^2)J_k]^{-\frac{1}{2}}\ . \tag{88}$$

These approximations for $\vec{n}(\vec{z})$, $< \vartheta >$, and P_{lim} can only be accurate if $|\alpha_n|$ is small.

In a ring with midplane symmetry, the one turn spin–orbit matrix $\underline{M}_{7\times 7}$ has a block structure with 2×2 matrix blocks and 2 dimensional zero and non–zero vectors,

$$\underline{M}_{7\times 7} = \begin{pmatrix} \underline{*} & \underline{0} & \underline{*} & \vec{0} \\ \underline{0} & \underline{*} & \underline{0} & \vec{0} \\ \underline{*} & \underline{0} & \underline{*} & \vec{0} \\ \vec{0}^T & \vec{*}^T & \vec{0}^T & * \end{pmatrix}\ . \tag{89}$$

The 6×6 dimensional phase space transport matrix has a chequer–board structure, since there is no coupling between vertical motion and the other two degrees of freedom in a midplane–symmetric ring; and $\vec{G}$ has only contributions from vertical motion since a spin with $\vec{S}_i = \vec{n}_0$ ($\alpha_i = 0$) is not deflected out of the vertical unless the particle flies through horizontal magnetic field components, which only happens for particles with a vertical oscillation amplitude. The opening angle $< \vartheta >$ and P_{lim} then only depend on the vertical action J_y.

3.1.2 Spin–Orbit–Coupling Integrals

Instead of computing the one turn matrix $M_{7\times7}$ as the product of spin–orbit transport matrices of individual elements or by concatenation of their spin transport quaternions, one can also solve the linearized equation of motion for α directly. To obtain simplified formulas, now the dreibein $[\vec{m}_0, \vec{l}_0, \vec{n}_0]$ is used which was introduced in section 2.2.6. The vectors $\vec{m}_0$ and $\vec{l}_0$ are perpendicular to $\vec{n}_0$, precess according to the T–BMT equation on the closed orbit, and are therefore related to $\vec{m}$ and $\vec{l}$ by a rotation around $\vec{n}_0$ with $\vec{m} + i\vec{l} = e^{i\nu_0(\theta-\theta_0)}(\vec{m}_0 + i\vec{l}_0)$, which was already derived in equation (57). Here it is assumed that the two dreibeins coincide at azimuth 0. Whereas the dreibein $[\vec{m}, \vec{l}, \vec{n}_0]$ constitutes a coordinate system which is 2π periodic in θ, $[\vec{m}_0, \vec{l}_0, \vec{n}_0]$ does not.

The precession vector for spins can be separated into a part for particles on the closed orbit and a part due to phase space amplitudes, $\vec{\Omega}(\vec{z}, \theta) = \vec{\Omega}_0(\theta) + \vec{\omega}(\vec{z}, \theta)$. The spin direction and the phase space dependent part $\vec{\omega}$ of the precession vector will be written in complex notation in the dreibein $[\vec{m}_0, \vec{l}_0, \vec{n}_0]$ as

$$\vec{S} = \mathrm{Re}\{\alpha_0\}\vec{m}_0 + \mathrm{Im}\{\alpha_0\}\vec{l}_0 + \vec{n}_0\sqrt{1+|\alpha_0|^2}\ , \tag{90}$$

$$\vec{\omega} = \mathrm{Re}\{\omega_0\}\vec{m}_0 + \mathrm{Im}\{\omega_0\}\vec{l}_0 + \vec{n}_0\omega_3\ . \tag{91}$$

Inserting this into the T–BMT equation (24), one obtains

$$\vec{\Omega} \times \vec{S} = \frac{d}{d\theta}\vec{S} = \mathrm{Re}\{\frac{d}{d\theta}\alpha_0\}\vec{m}_0 + \mathrm{Im}\{\frac{d}{d\theta}\alpha_0\}\vec{l}_0 + \vec{n}_0\frac{d}{d\theta}\sqrt{1-|\alpha_0|^2} + \vec{\Omega}_0 \times \vec{S}\ . \tag{92}$$

This leads to a differential equation for α_0,

$$\begin{aligned}\frac{d}{d\theta}\alpha_0 &= (\vec{\omega}\times\vec{S})\cdot(\vec{m}_0 + i\vec{l}_0) = \vec{\omega}\cdot[\vec{S}\times(\vec{m}_0 + i\vec{l}_0)] \\ &= \vec{\omega}\cdot[i\alpha_0\vec{n}_0 + \sqrt{1-|\alpha_0|^2}\ (\vec{l}_0 - i\vec{m}_0)] = -i\omega_0\sqrt{1-|\alpha_0|^2} + i\alpha_0\omega_3\ .\end{aligned} \tag{93}$$

Linearization with respect to $\vec{z}$ and α_0 leads to $\frac{d}{d\theta}\alpha_0 = -i\omega_0^1$, where the superscript signals the first–order expansion of $\omega(\vec{z}, \theta)$ with respect to $\vec{z}$. Since $\alpha = \alpha_0 e^{i\nu_0(\theta-\theta_0)}$, and since $\alpha_i = 0$ at azimuth θ_0 is transported to $\alpha_f = \vec{G}\cdot\vec{z}_i$ after one turn, one now obtains

$$\vec{G}\cdot\vec{z}_i = -ie^{i2\pi\nu_0}\int_{\theta_0}^{\theta_0+2\pi}\omega_0^1(\vec{z}(\theta),\theta)d\theta\ , \tag{94}$$

where the trajectory $\vec{z}(\theta)$ has started with $\vec{z}_i$ at azimuth θ_0.

In flat rings it is advantageous to use the comoving dreibein $[\vec{e}_x, \vec{e}_\theta, \vec{e}_y]$ introduced in section (2.1.2) with $\vec{e}_y$ vertical and with $\vec{e}_\theta$ parallel to the closed orbit. In such a ring, $\vec{n}_0$ and $\vec{\Omega}_0 = \Omega_0\vec{e}_y$ are vertical, and for a particle on the closed orbit, a spin has rotated by the angle $\Psi = \int_{\theta_0}^{\theta}\Omega_0 d\theta$ between azimuth θ_0 and θ. Therefore $\vec{m}_0 + i\vec{l}_0 = e^{-i\Psi}(\vec{e}_x + i\vec{e}_y)$ and $\omega_0^1 = e^{-i\Psi}(\omega_x^1 + i\omega_\theta^1)$.

In a midplane–symmetric ring, there are no skew elements or solenoids and horizontal components of $\vec{\omega}$ only occur when a particle oscillates vertically around the closed orbit. When linearizing in $\vec{z}$, these components are produced by the quadrupole focussing strength k ($k > 0$ for a horizontally focussing effect). The spin rotations in these fields are $(G\gamma + 1)$ larger than the orbit deflections created by the quadrupoles, and one obtains $\omega_0^1 d\theta = (G\gamma+1)e^{-i\Psi}ykdl$, where L is the circumference of the ring and $l = L\frac{\theta}{2\pi}$ is the path–length of the design trajectory. In terms of the

vertical betatron function β_y and the betatron phase Φ_y, one has $y = \sqrt{2J_y\beta_y}\cos(\Phi_y + \Phi_{yi})$. This has lead to the definition of the one turn spin–orbit–coupling integrals

$$I_y^{\pm} = -i(G\gamma + 1)\frac{1}{\sqrt{2}} \oint_{l_0}^{l_0+L} e^{i(-\Psi \pm \Phi_y)} \sqrt{\beta_y} k dl \ , \tag{95}$$

where $\Psi(\theta_0) = 0$ and the initial betatron phase is $\Phi_y(\theta_0) = 0$. When the initial phase space coordinate $\vec{z}_i$ has the vertical phase Φ_{yi} and the Courant–Snyder invariant [12] $2J_y$, equation (94) leads to

$$\vec{G} \cdot \vec{z}_i = e^{i2\pi\nu_0}(I_y^+ e^{i\Phi_{yi}} + I_y^- e^{-i\Phi_{yi}})\sqrt{J_y} \ . \tag{96}$$

With $\vec{z}_i = \sqrt{J_y}(\vec{v}_2^+ e^{i\Phi_{yi}} + \vec{v}_2^- e^{-i\Phi_{yi}})$ one obtains $\vec{G} \cdot \vec{v}_2^{\pm} = e^{i2\pi\nu_0} I_y^{\pm}$. Spin–orbit–coupling integrals are therefore useful for analyzing linear spin–orbit motion in the case of a midplane–symmetric ring.

In a general setting, where $\vec{\omega}^{\,1}(\vec{z}(\theta), \theta)$ not only has radial components, generalized spin–orbit–coupling integrals at θ_0 are defined as $I_k^{\pm} = e^{-i2\pi\nu_0}\vec{G} \cdot \vec{v}_k^{\pm}$. This brings them into close relation with the components $B_k^{\pm}$ of the $\vec{n}$–axis, which can be written as

$$\alpha_n = \sum_{k=1}^{3} \sqrt{J_k}(B_k^+ e^{i\Phi_k} + B_k^- e^{-i\Phi_k}) \ , \quad B_k^{\pm} = \frac{I_k^{\pm}}{e^{i2\pi(\pm Q_k - \nu_0)} - 1} \ . \tag{97}$$

So far eigenvectors of the one turn matrix have only been used at the initial azimuth θ_0. Now the eigenvectors $\vec{v}_k^{\pm}(\theta)$ of the one turn matrix at θ are needed which lead to the trajectory $\vec{z}(\theta)$ for a particle which started with the initial phase variables Φ_{ki} at azimuth 0,

$$\vec{z}(\theta) = \sum_{k=1}^{3} \sqrt{J_k}[\vec{v}_k^+(\theta) e^{i(Q_k(\theta-\theta_0)+\Phi_{ki})} - \vec{v}_k^-(\theta) e^{-i(Q_k(\theta-\theta_0)+\Phi_{ki})}] \ , \tag{98}$$

By inserting this into equation (94) and taking advantage of the linearity of ω_0^1, one obtains

$$I_k^{\pm} = e^{-i2\pi\nu_0}\vec{G} \cdot \vec{v}_k^{\pm} = -i \int_{\theta_0}^{\theta_0+2\pi} \vec{\omega}^1(\vec{v}_k^{\pm}(\theta), \theta) \cdot (\vec{m} + i\vec{l}) e^{i(\pm Q_k - \nu_0)(\theta-\theta_0)} d\theta \ . \tag{99}$$

One might be lead to think that $|I_2^+|^2 + |I_2^-|^2$ could be used as a quality factor for polarized proton synchrotrons. Due to the central importance of the invariant spin field for the acceleration process and for storage of polarized beams, it now becomes clear that the quality factor should in general rather be $\sum_{k=1}^{3}[\frac{|I_k^+|^2}{\sin^2(\pi(Q_k-\nu_0))} + \frac{|I_k^{\pm}|^2}{\sin^2(\pi(-Q_k-\nu_0))}]$. Close to intrinsic resonances where $\pm Q_k - \nu_0$ is integer for some k, the opening angle of the $\vec{n}$–axis diverges in linearized spin–orbit motion.

3.1.3 Restrictions of Linearized Spin–Orbit Motion

The approximation of linearized spin–orbit motion is no longer justified when P_{lim} is not close to 1, which happens close to intrinsic resonances in the figures 6. Linearized spin–orbit motion can be applied even when the resonances are not well separated, but when computing the average polarization of a polarized beam, $|\alpha_n|$ must be small enough to justify the underlying approximation. If $|\alpha_n| \leq 0.5$ is accepted, the average polarization computed with linearized spin–orbit motion is only trustworthy as long as it is above about 87%.

Figure 6 shows for DESY III (top) and for PETRA (bottom), that at most energies spin dynamics can be described well by linearized spin–orbit motion.

3.2 The Resonance Spectrum

The spin dynamics close to intrinsic resonances can be analyzed by Fourier expanding the field components $\vec{\omega}(\vec{z}, \theta)$ which perturb the spin of a particle that oscillates around the closed orbit.

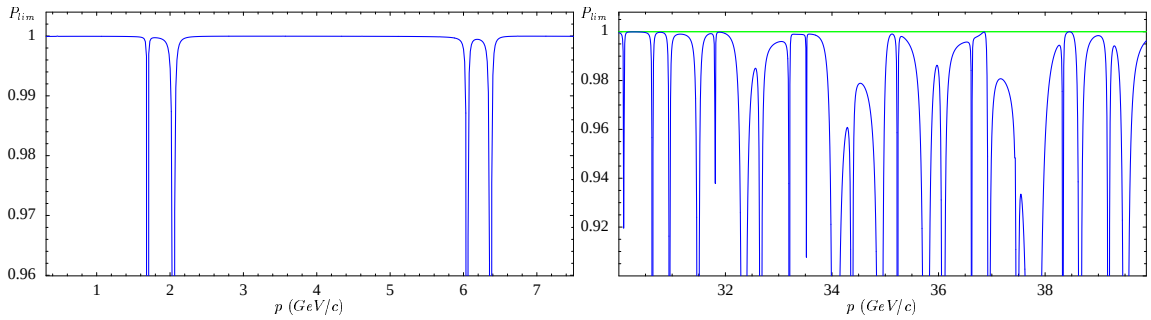

Figure 6: P_{lim} as approximated by linearized spin–orbit motion for DESY III (top) and for the high energy end of PETRA (bottom). The dips have been cut in order to magnify the interesting region where $|\alpha_n|$ is small.

For spins parallel to the rotation vector on the closed orbit $\vec{n}_0(\theta)$, only the components of $\vec{\omega}(\vec{z},\theta)$ which are perpendicular to $\vec{n}_0$ perturb the polarization.

As described in section 2.2.6, a depolarizing resonance occurs when a Fourier component of $\vec{\omega}(\vec{z}(\theta),\theta)$ rotates with the same frequency around $\vec{n}_0$ as the spins so that there is a strong perturbation. In the 2π periodic coordinate system constituted by $[\vec{m},\vec{l},\vec{n}_0]$, the Fourier component of $\omega = \vec{\omega}\cdot(\vec{m}+i\vec{l})$ for the frequency κ is computed by

$$\tilde{\epsilon}_\kappa = \lim_{N\to\infty}\frac{1}{2\pi N}\int_0^{2\pi N}\omega(\vec{z}(\theta),\theta)e^{-i\kappa\theta}d\theta\ . \tag{100}$$

A warning is needed. The picture of perturbing effects suggests that the beam is slowly depolarized after it has been injected with 100% polarization. In fact the spins get deflected from their initial polarization direction $\vec{n}_0$ during one turn, only because the $\vec{n}$–axis $\vec{n}(\vec{z})$ is tilted away from the closed orbit spin direction $\vec{n}_0$. If an ensemble of the spins had started parallel to their invariant spin field, no net deflection due to the perturbing fields would have occurred and no reduction of polarization would be noticed after one turn. However, since $\vec{n}(\vec{z})$ is tilted away from $\vec{n}_0$, the average polarization $P_{lim} = |<\vec{n}(\vec{z})>|$ for such an initial distribution is smaller than 1 to start with.

For each energy of the particle, there is in general a different Fourier spectrum of ω. Since at each energy the most important frequencies κ are those which are close to resonance with ν_0, it is customary to compute $\epsilon_{\nu_0(E)} = |\tilde{\epsilon}_{\nu_0(E)}|$, which is called the resonance strength, for all energies of the acceleration cycle. Obviously, $\epsilon_{\nu_0(E)}$ is zero, except when a Fourier frequency of $\omega(\vec{z}(\theta),\theta)$ at energy E is equal to $\nu_0(E)$. The resulting line spectrum of over E is called the depolarizing resonance spectrum of an accelerator.

For the proton synchrotrons DESY III and PETRA, these resonance strengths ϵ_{ν_0} are shown in the top figures 7 and 8. They were all computed for an oscillation amplitude of $\vec{z}(\theta)$ corresponding to the 2.5σ vertical emittance of 25πmm mrad.

It is possible to recover the first–order isolated resonance strength from the one turn spin–orbit transport matrix. For a spin which was initially parallel to $\vec{n}_0$, equation (93) yields

$$\alpha_0(\theta) \approx -i\int_0^\theta \omega_0 d\theta\ ,\quad \omega_0 = \vec{\omega}\cdot(\vec{m}_0+i\vec{l}_0) = e^{-i\nu_0\theta}\vec{\omega}\cdot(\vec{m}+i\vec{l})\ . \tag{101}$$

Comparing with equation (100), one can express the resonance strength by

$$\tilde{\epsilon}_{\nu_0} = i\lim_{N\to\infty}\frac{1}{2\pi N}\alpha_0(2\pi N)\ . \tag{102}$$

The resonance strength can therefore be computed from $\frac{1}{N}\underline{M}^N_{77}$ for large N. The computation becomes very efficient if one uses $\underline{M}^{2N}_{77} = (M^N_{77})^2$ iteratively.

The coordinate vectors $\vec{m}_0(2\pi)$ and $\vec{l}_0(2\pi)$ to which $\alpha_0(2\pi)$ refers have rotated by $2\pi\nu_0$, whereas the final spin coordinate α_f computed by $\underline{M}_{77}$ refers to the coordinate vectors $\vec{m}_0(0)$ and $\vec{l}_0(0)$. Therefore $\alpha_0(2\pi N) = \alpha_f \exp(-i2\pi N\nu_0)$. The resonance strength ϵ_{ν_0} can most easily be computed when the powers of the one turn matrix are evaluated in diagonal form using the diagonal matrix $\underline{\Lambda}$ with the elements $e^{\pm i2\pi Q_k}$. For ease of notation, $\breve{Q}_k$ is used with $\breve{Q}_{2k-1} = Q_k$ and $\breve{Q}_{2k} = -Q_k$,

$$\begin{aligned}
\tilde{\epsilon}_{\nu_0} &= i \lim_{N\to\infty} \frac{1}{2\pi N}\alpha_0(2\pi N) = i \lim_{N\to\infty} \frac{1}{2\pi N}(0, e^{-iN2\pi\nu_0})\begin{pmatrix} \underline{M} & \underline{0} \\ \vec{G}^T & e^{i2\pi\nu_0} \end{pmatrix}^N \begin{pmatrix} \vec{z} \\ 0 \end{pmatrix} \\
&= i \lim_{N\to\infty} e^{-iN2\pi\nu_0} \frac{1}{2\pi N} \sum_{j=0}^{N-1} [e^{i(N-j-1)2\pi\nu_0} \vec{G}^T \underline{A}^{-1} \underline{\Lambda}^j] \underline{A}\vec{z} \\
&= ie^{-i2\pi\nu_0} \sum_{k=1}^{6} G_l A^{-1}_{lk} A_{km} z_m \lim_{N\to\infty} \frac{1}{2\pi N} \sum_{j=0}^{N-1} e^{i2\pi j(\breve{Q}_k - \nu_0)} \qquad (103)
\end{aligned}$$

where one has to sum over equal indices l, and m. This formula shows that the resonance strength is always zero, except at a resonance condition $\nu_0 = \kappa = j_0 \pm Q_k$. At such a closed–orbit spin tune, the resonance strength is given by

$$\begin{aligned}
2\pi\epsilon_{\nu_0} &= |\vec{G}^T \underline{A}^{-1} \underline{\mathrm{diag}}(0...1...0) \underline{A}\vec{z}| = |\vec{G}^T \underline{A}^{-1} (0...\sqrt{J_k} e^{\pm i\Phi_k}...0)^T| \\
&= |\vec{G} \cdot \vec{v}_k^{\pm}| \sqrt{J_k} = |I_k^{\pm}| \sqrt{J_k} \qquad (104)
\end{aligned}$$

and $\tilde{\epsilon}_{\nu_0} = \epsilon_{\nu_0} e^{i2\pi(\pm\Phi_k - \nu_0)}$. The 1 in the diagonal matrix is in position $2k-1$ for $\nu_0 = j_0 + Q_k$ and at position $2k$ for $\nu_0 = j_0 - Q_k$. Here $\underline{A}^{-1}(0...\sqrt{J_k}e^{\pm i\Phi_k}...0)^T$ is the initial value for a phase space trajectory which has only Fourier components with frequencies $\pm Q_k$ plus integers and the eigenvector $\vec{v}_k^{\pm}$ of $\underline{M}$ has been used. The infinite Fourier integral in equation (100) has been reduced to the scalar product between the bottom row vector of $\underline{M}_{77}$ and an eigenvector of $\underline{M}$ which happens to equal the absolute value of the spin–orbit–coupling integral in equation (97). This very simple formula [39] is used in the program SPRINT [37, 33].

After the first–order resonance strength for a frequency κ has been computed, one can investigate the influence of only the one corresponding Fourier contribution of ω to the spin motion. The resulting single resonance model (SRM) has been described in section 2.2.6.

3.2.1 Limitations of the SRM

Approximating the spin motion by the SRM is only accurate if the resonances are well separated so that one Fourier harmonic of ω dominates the dynamics. When a ring is not flat and has no exact super–periodicity, the first–order resonances appear when the spin tune comes close to $j_0 \pm Q_k$, where the tunes Q_k of all three degrees of motion can appear. HERA–p is not flat, but after the installation of flattening snakes, the first–order spin motion is very similar to that of a flat ring, where only resonances due to vertical motion appear. With a vertical orbit tune of approximately $\frac{1}{3}$ in HERA–p, the variation of ν_0 between resonances is $\frac{1}{3}$ or $\frac{2}{3}$. The resonance strength is related to the width of the resonance as shown in section 2.2.6; and to justify a single resonance approach, the resonance strength of two neighboring resonances should therefore be significantly less than $\frac{1}{3}$.

In linearized spin–orbit motion, the opening angle of the invariant spin field is approximately given by equation (87). In figure 7 the peaks in the resonance strength (top) are located exactly at the peaks of the big opening angles computed with the linearized approach (bottom); furthermore the widths of the peaks in opening angle are correlated with the resonance strengths. The resonances are well separated and in DESY III, first–order theories for analyzing polarization dynamics along with classical means of controlling depolarizing effects [40] are therefore applicable.

The corresponding figure for PETRA shows again that large opening angles of linearized spin–orbit motion are correlated with large resonance strength. However, the first–order resonances are getting so close at the high energy end of 39GeV that several pairs of resonances are close to overlapping. The resonance strengths are still far away from PETRA's fractional vertical tune

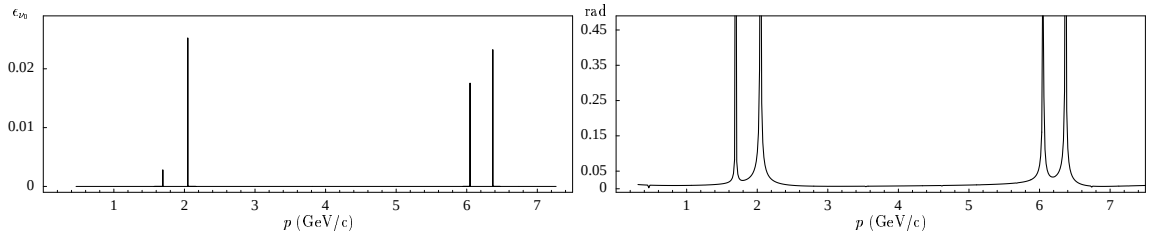

Figure 7: Resonance strength (top) and opening angles of linearized spin–orbit motion (bottom) for particles with a normalized vertical amplitude of 25πmm mrad in DESY III. The number of resonances is very low due to a super–periodicity 8.

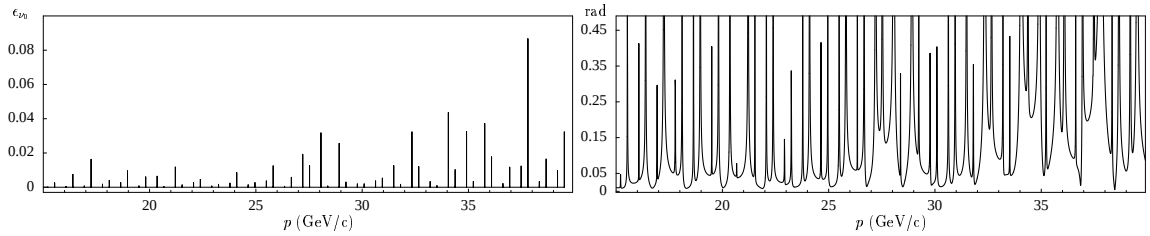

Figure 8: Resonance strength (top) and opening angles of linearized spin–orbit motion (bottom) for particles with normalized vertical amplitude of 25πmm mrad in PETRA.

of about 0.2 and therefore also in this energy regime classical means of controlling depolarizing first–order resonances can be applied.

The average polarization computed with either of these two models, linearized spin–orbit motion or the single resonance model with first order resonances, is in any case only accurate if there are only effects which are dominated by first–order resonances. Effects which are not related to first–order resonances cannot be simulated by a first–order resonance strength or by linearized spin–orbit motion and therefore the first–order theories cannot be used to decide whether non–first–order effects are small or not. In general, therefore, a higher–order extension is needed to decide about the validity of the first–order theories. Dealing with higher–order effects is outside the scope of this article but has been dealt with in detail in [1, 6].

References

[1] G. H. Hoffstaetter. *Aspects of the invariant spin field for high energy polarized proton beams.* Habilitation, Darmstadt Univ., 2000.

[2] D. P. Barber and A. De Roeck, editors. *Proceedings of Polarized Protons at High Energies*, DESY–Proceedings–1999–03, 1999.

[3] A. Belov. Atomic beam type polarized ion sources. In [2].

[4] A. Zelenski. Optically pumped polarized ion sources (OPPIS). In [2].

[5] G. H. Hoffstaetter, M. Vogt, and D. P. Barber. Higher–order effects in polarized proton dynamics. *Phys. Rev. ST–AB*, 2(11):1–15, 1999.

[6] M. Vogt: Bounds on the maximum attainable equilibrium spin polarization of protons in HERA. *Dissertation, Universität Hamburg*, DESY–THESIS–2000–054 (December 2000)

[7] A. Lehrach. Calculation of the invariant spin field by adiabatically blowing up the beam with an rf dipole. In [2].

[8] L. H. Thomas. The kinematics of an electron with an axis. *Phil. Mag.*, 3(13):1–20, 1927.

[9] V. Bargmann, L. Michel, and V. L. Telegdi. Precession of the polarization of particles moving in a homogeneous electro–magnetic field. *Phys. Rev. Lett.* 2(10):435–436, 1959.

[10] H. Rose. Hamiltonian magnetic optics. *Nucl. Instr. Meth.* A258:374–401, 1987.

[11] G. H. Hoffstaetter. Nonlinear dependence of synchrotron radiation on beam parameters. In *Proceedings of PAC 94, Dallas/TX*, 1995.

[12] E. D. Courant and H. S. Snyder. Theory of the alternating–gradient synchrotron. *Ann. Phys.* 3:1–48, 1958.

[13] S. Y. Lee. *Spin dynamics and Snakes in synchrotrons.* World Scientific, 1997.

[14] Bryan W. Montague. Polarized beams in high energy storage rings. *Physics Reports*, 113(1):1–96, 1984.

[15] G. H. Hoffstaetter, H. S. Dumas and J. A. Ellison: Adiabatic invariants for spin-orbit motion. In *Proceedings to EPAC02*, Paris and DESY-M-02–01 (2002)

[16] H. Huang, L. Ahrens, J. G. Alessi, M. Beddo, K. A. Brown, *et al.* Preservation of proton polarization by a partial Siberian Snake. *Phys. Rev. Lett.* 73:2982–2985, 1994.

[17] B. B. Blinov, C. M. Chu, E. D. Courant, D. A. Crandell, *et al.* First test of a partial Siberian Snake during polarized beam acceleration. *Phys. Rev. Lett.* 73(12):1621–1623, 1994.

[18] T. Roser. RHIC spin program: machine aspects and recent progress. In [2].

[19] M. Bai, L. Ahrens, J. Alessi, K. Brown, G. Bunce, P. Cameron, *et al.* Overcoming intrinsic spin resonances with an rf dipole. *Phys. Rev. Lett.* 80(21):4673–4676, 1998.

[20] Ya. S. Derbenev, A. M. Kondratenko, S. I. Serednyakov, *et al.* Radiative polarization: obtaining, control, using. *Particle Accelerators*, 8:115–126, 1978.

[21] A. D. Krisch, S. R. Mane, R. S. Raymond, T. Roser, *et al.* First test of the Siberian Snake magnet arrangement to overcome depolarizing resonances in a circular accelerator. *Phys. Rev. Lett.* 63(11):1137–1140, 1989.

[22] V. Anferov. Siberian Snake design for a high energy ring. In [2].

[23] G. H. Hoffstaetter. Snake matching. In UM–HE 99–05, University of Michigan, 1999.

[24] G. H. Hoffstaetter. Matching of Siberian Snakes. AIP Proceedings, Ann Arbor, November 2002.

[25] V. I. Ptitsin and Yu. M. Shatunov. Helical spin rotators and snakes. In *Proc. 3. Workshop on Siberian Snakes and Spin Rotators*, BNL-52453:15, Brookhaven, 1994.

[26] D. P. Barber, G. H. Hoffstaetter, and M. Vogt. Polarized protons in HERA. In *Future Physics at HERA*, volume 2, p. 1205–1216. DESY, 1996.

[27] G. H. Hoffstaetter. Polarized protons in HERA. In DESY–96–05, 1996.

[28] G. H. Hoffstaetter. Polarized protons at HERA: the accelerator issues. In [2].

[29] S. R. Mane. Electron-spin polarization in high energy storage rings. II. Evaluation of the equilibrium polarization. *Phys. Rev.* A(36):120–130, 1987.

[30] A. W. Chao. Polarization of a stored electron beam. In *Conference proceedings 87.* US Particle Accelerator School, Batavia, USA, American Institute of Physics, 1981.

[31] H. Mais and G. Ripken. Theory of spin–orbit motion in electron–positron storage rings. DESY–83–062, 1983.

[32] K. Abragam. *The principles of nuclear magnetism.* Clarendon, 1961.

[33] K. Heinemann and G. H. Hoffstaetter. A tracking algorithm for the stable spin polarization field in storage rings using stroboscopic averaging. *Phys. Rev.* E 54:4240–4255, 1996.

[34] M. Froissart and R. Stora. Depolarisation d'un faisceau de protons polarises dans un synchrotron. *Nucl. Instr. Meth.*, 7:297–305, 1960.

[35] E. D. Courant and R. D. Ruth. The acceleration of polarized protons in circular accelerators. BNL–51270 and UC–28 and ISA–80–5, 1980.

[36] D. P. Barber, J. Kewisch, G. Ripken, R. Rossmanith, and R. Schmidt. A solenoid spin rotator for large electron storage rings. *Particle Accelerators*, 17:243–262, 1985.

[37] G. H. Hoffstaetter and M. Vogt. Sprint users guide and reference manual. DESY, 1999.

[38] D. P. Barber and G. Ripken. Precession of the polarization of particles moving in a homogeneous electro–magnetic field. In A. W. Chao and M. Tigner, editors. *Handbook of accelerator physics and engineering* World Scientific, 1999.

[39] G. H. Hoffstaetter. Polarized protons in HERA. In *Proceedings of NUCLEON99, Nucl. Phys.* A, 1999.

[40] SPIN Collaboration and the DESY Polarization Team. Acceleration of polarized protons to 820 GeV at HERA. UM–HE–96–20, University of Michigan, 1996.