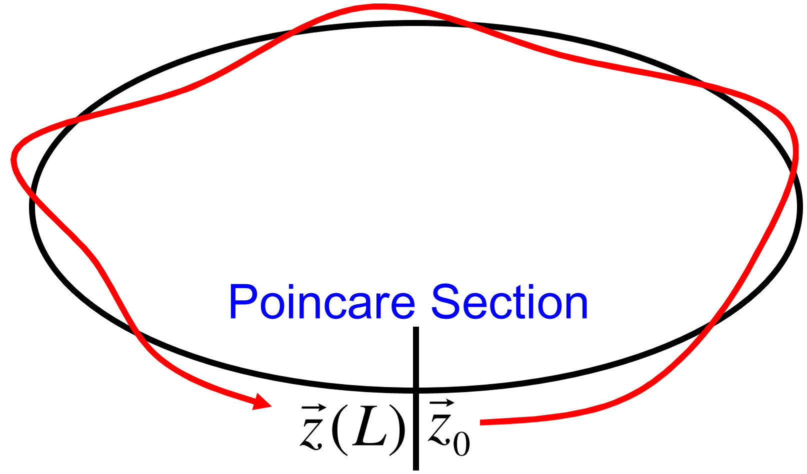
Tune shift in a ring due to quadrupole error

CHESS & LEPP

Average phase advance change per turn:

$$\overline{\Delta\psi} = \tfrac{1}{2}\Delta k l(\hat{s})\overline{\hat{\beta}} = \tfrac{1}{2}\Delta k l(\hat{s})\beta_0$$

Tune change: $\Delta\nu = \tfrac{1}{4\pi}\Delta k l(\hat{s})\beta_0$



$$\cos(\mu+\Delta\mu) \approx \cos\mu - \Delta\mu\sin\mu =$$

$$\tfrac{1}{2}\mathrm{Tr}\left[\begin{pmatrix} 0 & 0 \\ -\Delta k l(\hat{s}) & 0 \end{pmatrix}\begin{pmatrix} \cos\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu \end{pmatrix}\right] = \cos\mu - \tfrac{1}{2}\Delta k l(\hat{s})\beta\sin\mu$$

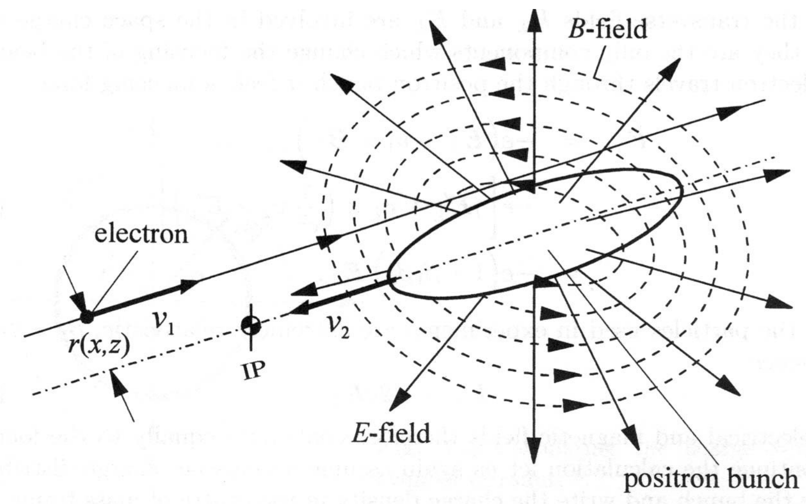
Oscillation frequencies can be measured relatively easily and accurately.

Measurement of beta function: Change k and measure tune.

The beam-beam tune shift

The force that acts from one beam to the other during collisions is focusing or defocusing in both planes for small distances.

For large distances it is very nonlinear.



The effects of E and B forces add. Whereas they subtract for co-moving particles.

$$\Delta\nu_x^{(1)} = \frac{r_{\mathrm{cl}}^{(1)} N_{\mathrm{cpb}}^{(2)}}{2\pi} \frac{\beta_x^{(1)}}{\sigma_x^{(2)}(\sigma_x^{(2)} + \sigma_y^{(2)})}$$

The beam-beam force

$$\left.\begin{aligned}\rho_{\text{lab}} &= \frac{1}{2\pi\sigma_x\sigma_y} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} \rho_{\text{lab}\,z}(z+v^{(2)}t) \\ \vec{j}_{\text{lab}} &= \vec{\beta} c \rho_{\text{lab}\,z}\end{aligned}\right\}\left\{\begin{aligned}\rho_{\text{rest}} &= \frac{1}{2\pi\sigma_x\sigma_y} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} \rho_z(z) \\ \vec{j}_{\text{rest}} &= 0\end{aligned}\right.$$

$$E_x(x,y,z) = \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}\int \frac{x-x_0}{(x-x_0)^2+(y-y_0)^2} e^{-(\frac{x_0^2}{2\sigma_x^2}+\frac{y_0^2}{2\sigma_y^2})} dx_0 dy_0$$

$$\approx \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}\int \frac{u}{u^2+v^2} e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int \frac{1}{u^2+v^2} e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int [\int_0^\infty e^{-t(u^2+v^2)} dt] e^{-(\frac{(x-u)^2}{2\sigma_x^2}+\frac{(y-v)^2}{2\sigma_y^2})} dudv$$

$$= \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{2\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(u^2(\frac{1}{2\sigma_x^2}+t)-\frac{2xu}{2\sigma_x^2}+v^2(\frac{1}{2\sigma_y^2}+t)-\frac{2yv}{2\sigma_y^2})} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} dudvdt$$

The beam-beam force

$$E_x(x,y,z) \approx \frac{Q\rho_z}{4\pi\varepsilon_0}\frac{1}{\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(u^2(\frac{1}{2\sigma_x^2}+t)-\frac{2xu}{2\sigma_x^2}+v^2(\frac{1}{2\sigma_y^2}+t)-\frac{2yv}{2\sigma_y^2})} e^{-(\frac{x^2}{2\sigma_x^2}+\frac{y^2}{2\sigma_y^2})} dudvdt$$

$$= \frac{Q\rho_z}{4\pi\varepsilon_0}\frac{1}{\pi\sigma_x\sigma_y}(\sigma_x^2\partial_x + x)\int e^{-(\tilde{u}^2(\frac{1}{2\sigma_x^2}+t)+\tilde{v}^2(\frac{1}{2\sigma_y^2}+t))} e^{-(\frac{x^2}{2\sigma_x^2}-\frac{x^2}{4\sigma_x^4(\frac{1}{2\sigma_x^2}+t)}+\frac{y^2}{2\sigma_y^2}-\frac{y^2}{4\sigma_y^4(\frac{1}{2\sigma_y^2}+t)})} dudvdt$$

$$= \frac{Q\rho_z}{4\pi\varepsilon_0}2(\sigma_x^2\partial_x + x)\int_0^\infty \frac{e^{-(\frac{x^2}{2\sigma_x^2+\frac{1}{t}}+\frac{y^2}{2\sigma_y^2+\frac{1}{t}})}}{\sqrt{2\sigma_x^2+\frac{1}{t}}\sqrt{2\sigma_y^2+\frac{1}{t}}}\frac{dt}{t}$$

$$= -\frac{Q\rho_z}{4\pi\varepsilon_0}\partial_x\int_0^\infty \frac{e^{-(\frac{x^2}{2\sigma_x^2+q}+\frac{y^2}{2\sigma_y^2+q})}}{\sqrt{2\sigma_x^2+q}\sqrt{2\sigma_y^2+q}}dq = -\partial_x U$$

$$\approx \frac{Q\rho_z}{2\pi\varepsilon_0}x\int_0^\infty \frac{1}{\sqrt{2\sigma_x^2+q}^3\sqrt{2\sigma_y^2+q}}dq = \frac{Q\rho_z}{2\pi\varepsilon_0}\frac{1}{\sigma_x(\sigma_x+\sigma_y)}x$$