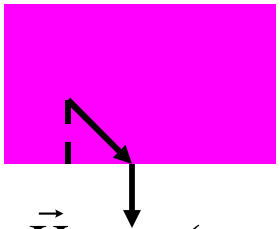
Surfaces of Equal Potential



$$\vec{B}_\perp(\text{out}) = \vec{B}_\perp(\text{in})$$

$$\vec{H}_{\text{parallel}}(\text{out}) = \vec{H}_{\text{parallel}}(\text{in})$$

$$\vec{B}_{\text{parallel}}(\text{out}) = \frac{1}{\mu_r}\vec{B}_{\text{parallel}}(in)$$

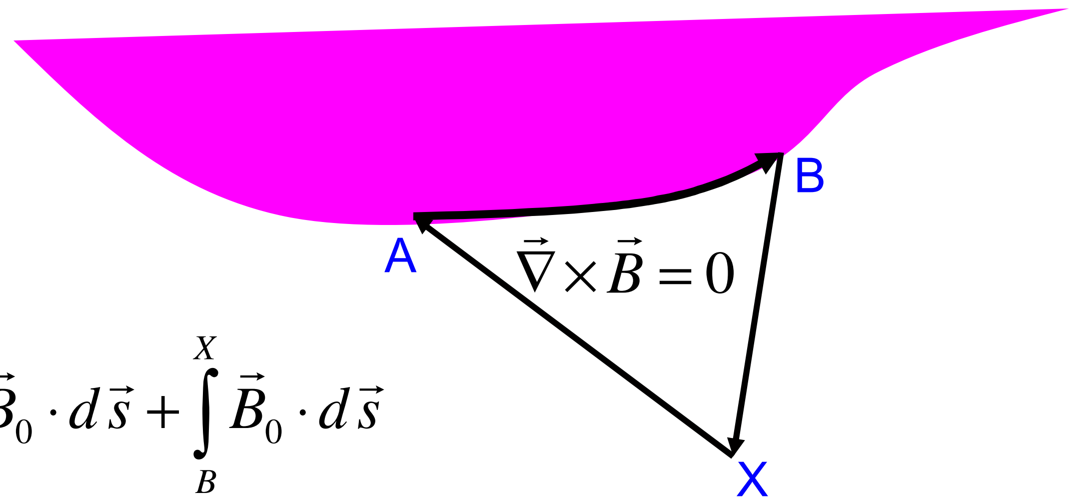
$$\vec{B}(\vec{\text{r}}) = -\vec{\nabla}\Psi(\vec{r})$$

$$0 = \oint \vec{B}\cdot d\vec{s} = \int_X^A \vec{B}_0\cdot d\vec{s} + \int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$= \int_X^A \vec{B}_0\cdot d\vec{s} + \frac{1}{\mu_r}\int_A^B \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s}$$

$$\approx \int_X^A \vec{B}_0\cdot d\vec{s} + \int_B^X \vec{B}_0\cdot d\vec{s} = \Psi(A) - \Psi(B)$$

For large permeability, H(out) is perpendicular to the surface.



For highly permeable materials (like iron) surfaces have a constant potential.

Green's Theorem

$$\vec{\nabla}^2\psi = 0$$

Green function:

$$\vec{\nabla}_0^2 G(\vec{r},\vec{r}_0) = \delta(\vec{r}-\vec{r}_0)$$

$$\begin{aligned}
\psi(\vec{r}) &= \int_V \psi(\vec{r}_0)\delta(\vec{r}-\vec{r}_0)\, d^3\vec{r}_0 \\
&= \int_V \left[\psi(\vec{r}_0)\vec{\nabla}_0^2 G - G\vec{\nabla}_0^2\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\
&= \int_V \vec{\nabla}_0\left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right] d^3\vec{r}_0 \\
&= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G - G\vec{\nabla}_0\psi(\vec{r}_0)\right]\cdot d^2\vec{r}_0 \\
&= \int_{\partial V} \left[\psi(\vec{r}_0)\vec{\nabla}_0 G + \vec{B}(\vec{r}_0) G\right]\cdot d^2\vec{r}_0
\end{aligned}$$

Knowledge of the field and the scalar magnetic potential on a closed surface inside a magnet determines the magnetic field for the complete volume which is enclosed.

Potential Expansion

If field data in a plane (for example the midplane of a cyclotron or of a beam line magnet) is known, the complete filed is determined:

$$\psi(x,y,z)=\sum_{n=0}^{\infty} b_n(x,z)\,y^n \quad\Rightarrow\quad \vec{B}(x,0,z)=-\begin{pmatrix}\partial_x b_0(x,z)\\ b_1(x,z)\\ \partial_z b_0(x,z)\end{pmatrix}$$

$$0=\vec{\nabla}^2\psi=\sum_{n=0}^{\infty}(\partial_x^2+\partial_z^2)\,b_n\ y^n+\sum_{n=2}^{\infty} n(n-1)\,b_n\ y^{n-2}$$

$$=\sum_{n=0}^{\infty}\left[(\partial_x^2+\partial_z^2)\,b_n+(n+2)(n+1)\,b_{n+2}\right]y^n$$

$$b_{n+2}(x,z)=-\frac{1}{(n+2)(n+1)}(\partial_x^2+\partial_y^2)\,b_n(x,z)$$

Data of the magnetic field in the plane y=0 is used to determine $b_0(x,z)$ and $b_1(x,z)$.

Complex Potentials

CHESS & LEPP

$$w = x + iy \quad , \quad \overline{w} = x - iy$$

$$\partial_x = \partial_w + \partial_{\overline{w}} \, , \quad \partial_y = i\partial_w - i\partial_{\overline{w}} = i(\partial_w - \partial_{\overline{w}})$$

$$\vec{\nabla}^2 = \partial_x^2 + \partial_y^2 + \partial_z^2 = (\partial_w + \partial_{\overline{w}})^2 - (\partial_w - \partial_{\overline{w}})^2 + \partial_z^2 = 4\partial_w \partial_{\overline{w}} + \partial_z^2$$

$$\psi = \mathrm{Im}\{ \sum_{\nu,\lambda=0}^{\infty} a_{\nu\lambda}(z) \cdot (w\overline{w})^{\lambda} \overline{w}^{\nu} \}$$

$$\vec{\nabla}^2 \psi = \mathrm{Im}\{ \sum_{\nu=0,\lambda=1}^{\infty} 4 a_{\nu\lambda} (\lambda + \nu) \lambda (w\overline{w})^{\lambda-1} \overline{w}^{\nu} + \sum_{\nu=0,\lambda=0}^{\infty} a_{\nu\lambda}'' (w\overline{w})^{\lambda} \overline{w}^{\nu} \}$$

$$= \mathrm{Im}\{ \sum_{\nu,\lambda=0}^{\infty} [4(\lambda + 1 + \nu)(\lambda + 1) a_{\nu\lambda+1} + a_{\nu\lambda}''] (w\overline{w})^{\lambda} \overline{w}^{\nu} \} = 0$$

Iteration equation: $a_{\nu\lambda+1} = \dfrac{-1}{4(\lambda+1+\nu)(\lambda+1)} a_{\nu\lambda}'' \quad , \quad a_{\nu 0} = \Psi_{\nu}(z)$

The functions $\Psi_\nu(z)$ along a line determine the complete field inside a magnet.

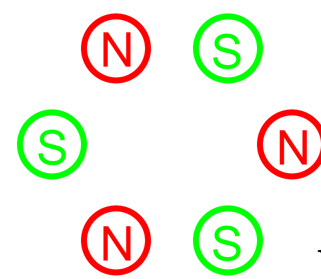
Multipole Coefficients

$\Psi_\nu(z)$ are called the z-dependent multipole coefficients

$$\psi(x,y,z) = \mathrm{Im}\{\sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{w\overline{w}}{4}\right)^\lambda \overline{w}^\nu \Psi_\nu^{[2\lambda]}(z)\}$$

$$\psi(r,\varphi,z) = \sum_{\nu,\lambda=0}^{\infty} \frac{(-1)^\lambda \nu!}{(\lambda+\nu)!\lambda!}\left(\frac{r}{2}\right)^{2\lambda} r^\nu\, \mathrm{Im}\{\Psi_\nu^{[2\lambda]}(z) e^{-i\nu\varphi}\}$$

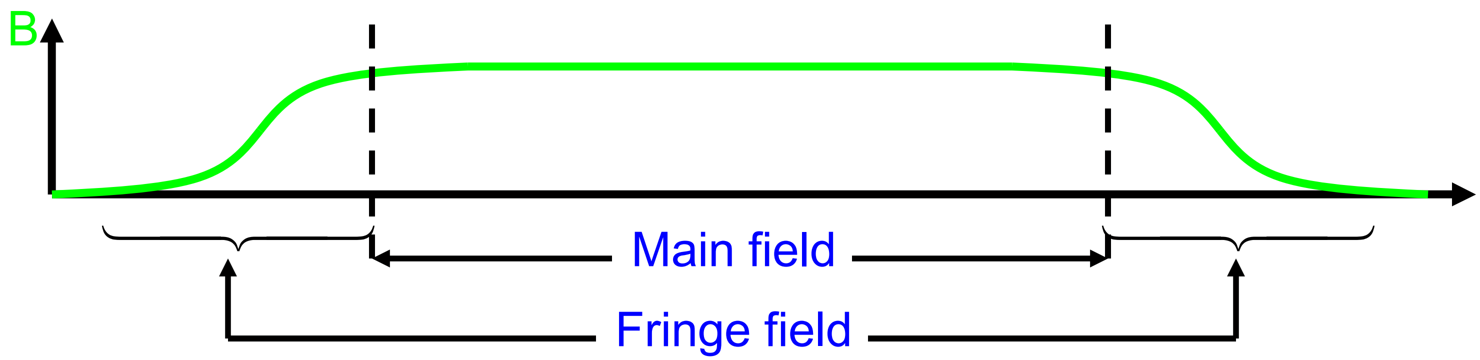
The index ν describes C_ν Symmetry
around the z-axis $\vec{e}_z$
due to a sign change after $\Delta\varphi = \dfrac{\pi}{\nu}$



$\nu = 3$

Fringe Fields and Main Fields

CHESS & LEPP



Only the fringe field region has terms with $\lambda \neq 0$ and $\partial_z^2 \psi \neq 0$

Main fields in accelerator physics: $\lambda = 0\,,\quad \partial_z^2 \psi = 0$

$$\Psi_\nu = \begin{cases} e^{i\nu\vartheta_\nu} |\Psi_\nu| & \text{for} \quad \nu \neq 0 \\ i \;\; |\Psi_0| & \text{for} \quad \nu = 0 \end{cases}$$

$$\psi(r,\varphi) = \sum_{\nu=1}^{\infty} r^\nu |\Psi_\nu| \,\mathrm{Im}\{e^{-i\nu(\varphi-\vartheta_\nu)}\} + |\Psi_0|$$

Main Field Potential

Main field potential: $$\psi = |\Psi_0| - \sum_{\nu=1}^{\infty} r^{\nu} |\Psi_{\nu}| \sin[\nu(\varphi - \vartheta_{\nu})]$$

The isolated multipole: $$\psi = -r^{\nu} |\Psi_{\nu}| \sin(\nu\varphi)$$

Where the rotation ϑ_{ν} of the coordinate system is set to 0

The potentials produced by different multipole components Ψ_{ν} have

a) Different rotation symmetry C_{ν}

b) Different radial dependence r^{ν}