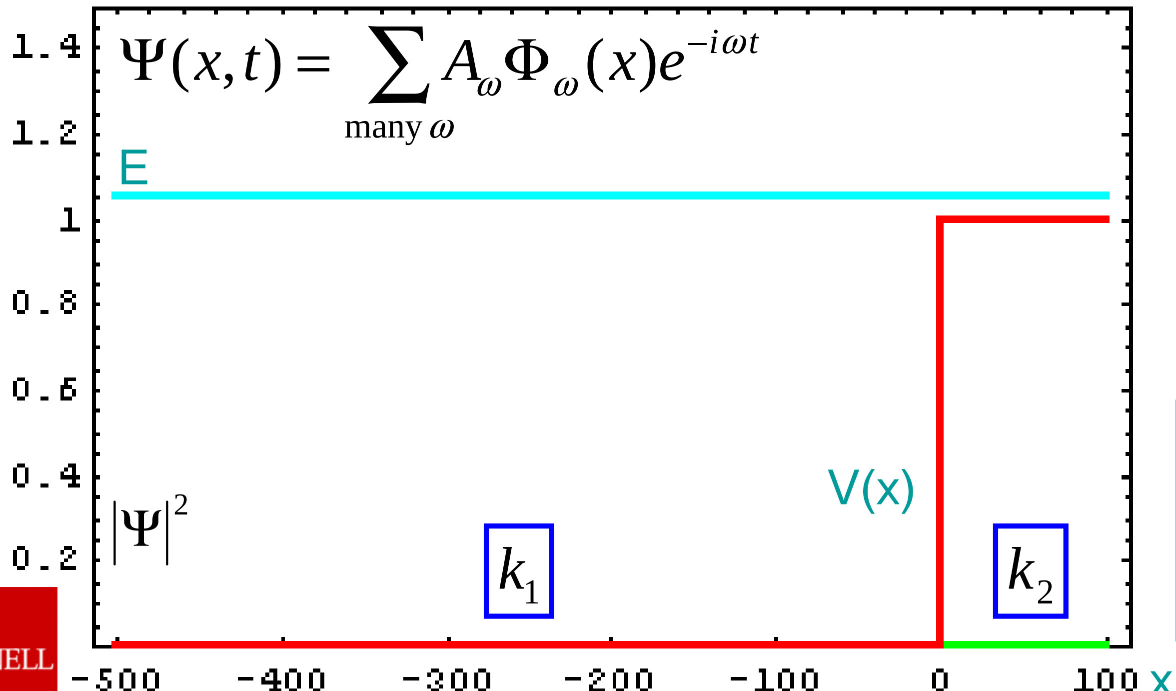
Potential step up

04/08/2005

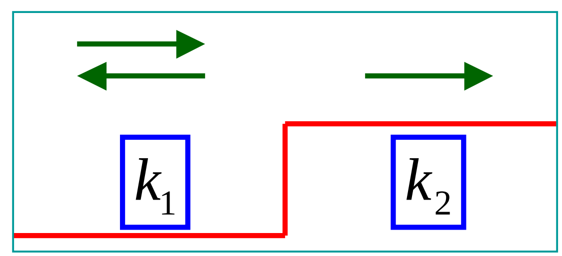
$$\Phi_\omega(x)e^{-i\omega t} = \begin{cases} Ae^{i(k_1 x - \omega t)} + Be^{i(-k_1 x - \omega t)} & \text{for } x < 0, \quad k_1 = \sqrt{\frac{2m}{\hbar^2}(\hbar\omega - V_1)} \\ Ce^{i(k_2 x - \omega t)} & \text{for } x \geq 0, \quad k_2 = \sqrt{\frac{2m}{\hbar^2}(\hbar\omega - V_2)} \end{cases}$$

$$\left.\begin{array}{l} \Phi_\omega(0_-) = \Phi_\omega(0_+) \rightarrow k_1 A + k_1 B = k_1 C \\ \frac{\partial}{\partial x}\Phi_\omega(0_-) = \frac{\partial}{\partial x}\Phi_\omega(0_+) \rightarrow k_1 A - k_1 B = k_2 C \end{array}\right\} \begin{array}{l} A = \frac{k_1 + k_2}{2k_1} C \\ B = \frac{k_1 - k_2}{2k_1} C \end{array}$$



Scattering occurs at abrupt changes of the potential.

Considered waves:



Georg.Hoffstaetter@Cornell.edu

Potential barrier

04/08/2005

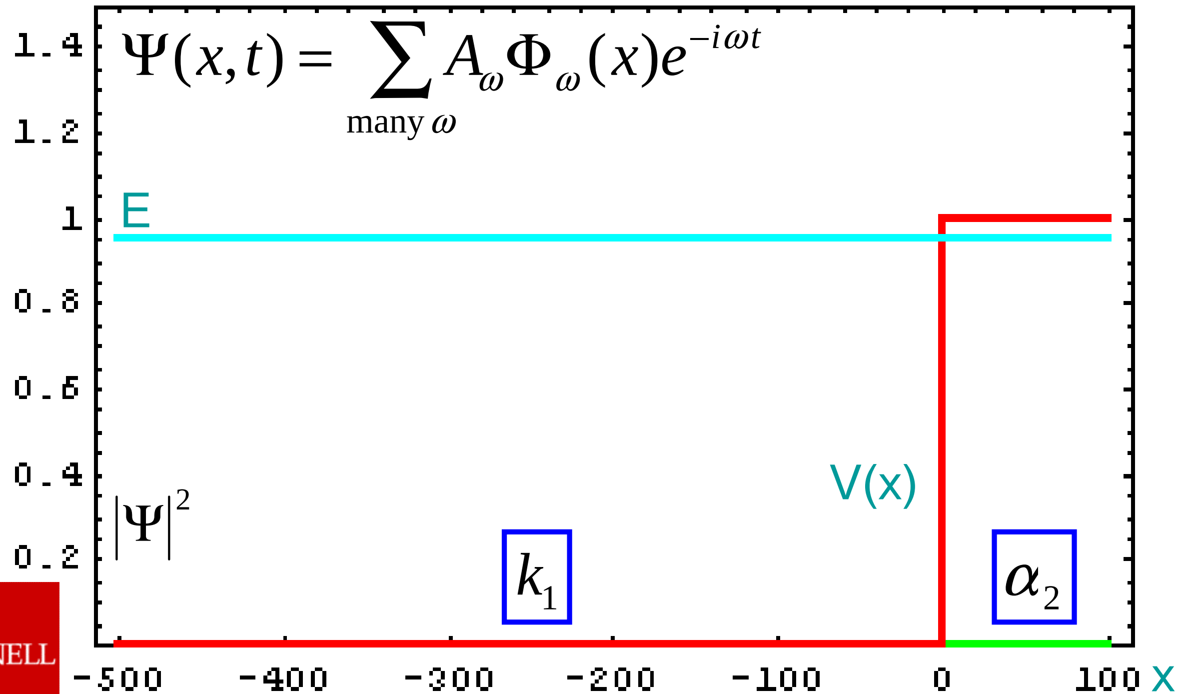
$$\Phi_\omega(x)e^{-i\omega t} = e^{-i\omega t}\begin{cases} Ae^{ik_1 x} + Be^{-ik_1 x} & \text{for } x<0, \quad k_1 = \sqrt{\frac{2m}{\hbar^2}(\hbar\omega - V_1)} \\ Ce^{-\alpha_2 x} & \text{for } x \ge 0, \quad \alpha_2 = \sqrt{\frac{2m}{\hbar^2}(V_2 - \hbar\omega)} = -ik_2 \end{cases}$$

$$\Phi_\omega(0_-) = \Phi_\omega(0_+) \rightarrow k_1 A + k_1 B = k_1 C$$

$$\frac{\partial}{\partial x}\Phi_\omega(0_-) = \frac{\partial}{\partial x}\Phi_\omega(0_+) \rightarrow k_1 A - k_1 B = i\alpha_2 C$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \begin{cases} A = \frac{k_1 + i\alpha_2}{2k_1} C \\ B = \frac{k_1 - i\alpha_2}{2k_1} C \end{cases}$$

$$\Psi(x,t) = \sum_{\text{many } \omega} A_\omega \Phi_\omega(x) e^{-i\omega t}$$



Example: The boundary between a metal and a vacuum.

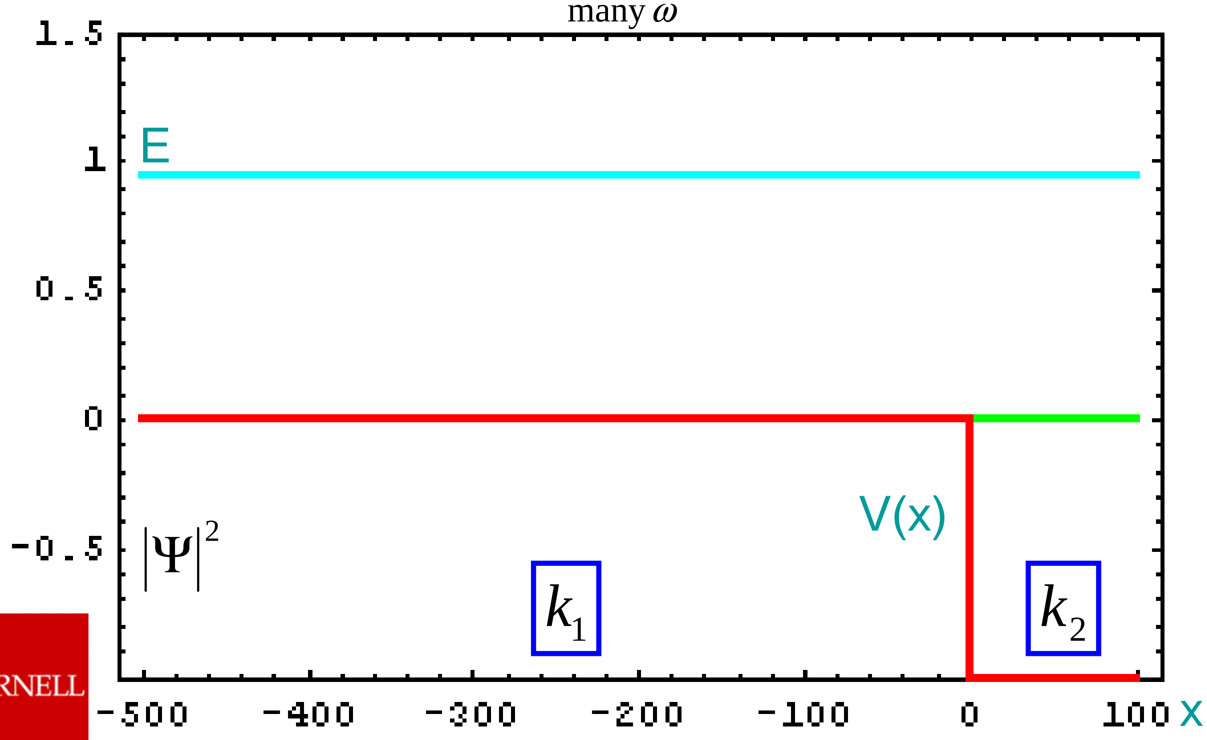
|A|=|B| means that every wave is completely reflected, even though it penetrates into the classically forbidden region.

Georg.Hoffstaetter@Cornell.edu

Potential step down

$$\Phi_\omega(x) = C\begin{cases} \frac{k_1+k_2}{2k_1} e^{ik_1 x} + \frac{k_1-k_2}{2k_1} e^{-ik_1 x} & \text{for } x<0 \\ e^{ik_2 x} & \text{for } x \geq 0 \end{cases}$$

$$\Psi(x,t) = \sum_{\text{many }\omega} A_\omega \Phi_\omega(x) e^{-i\omega t}$$

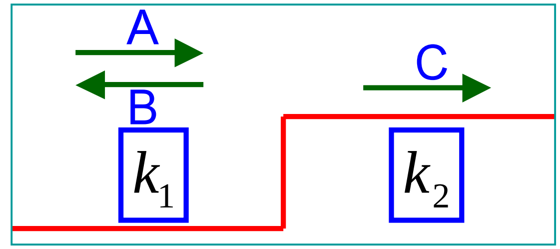


Scattering occurs even when the potential step goes down, which classically would not lead to a reflection for any energy.

Georg.Hoffstaetter@Cornell.edu

Probability density and probability current

Why is, at the boundary at x=0, the sum of incoming $|A|^2$ probability densities not equal to the sum of outgoing probability densities $|B|^2 + |C|^2$?



Because the incoming and outgoing currents have to match:

$$|A|^2 v_{group1} = |B|^2 v_{group1} + |C|^2 v_{group2}$$

$$|A|^2 k_1 = |B|^2 k_1 + |C|^2 k_2$$

$$\left(\frac{k_1+k_2}{2k_1}\right)^2 k_1 = \left(\frac{k_1-k_2}{2k_1}\right)^2 k_1 + 1 \cdot k_2$$ ✓

Reflection coefficient: $R = |B|^2 / |A|^2$

Transmission coefficient: $T = 1 - R = (|C|^2 k_2)/(|A|^2 k_1)$

$$R = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2$$

Georg.Hoffstaetter@Cornell.edu

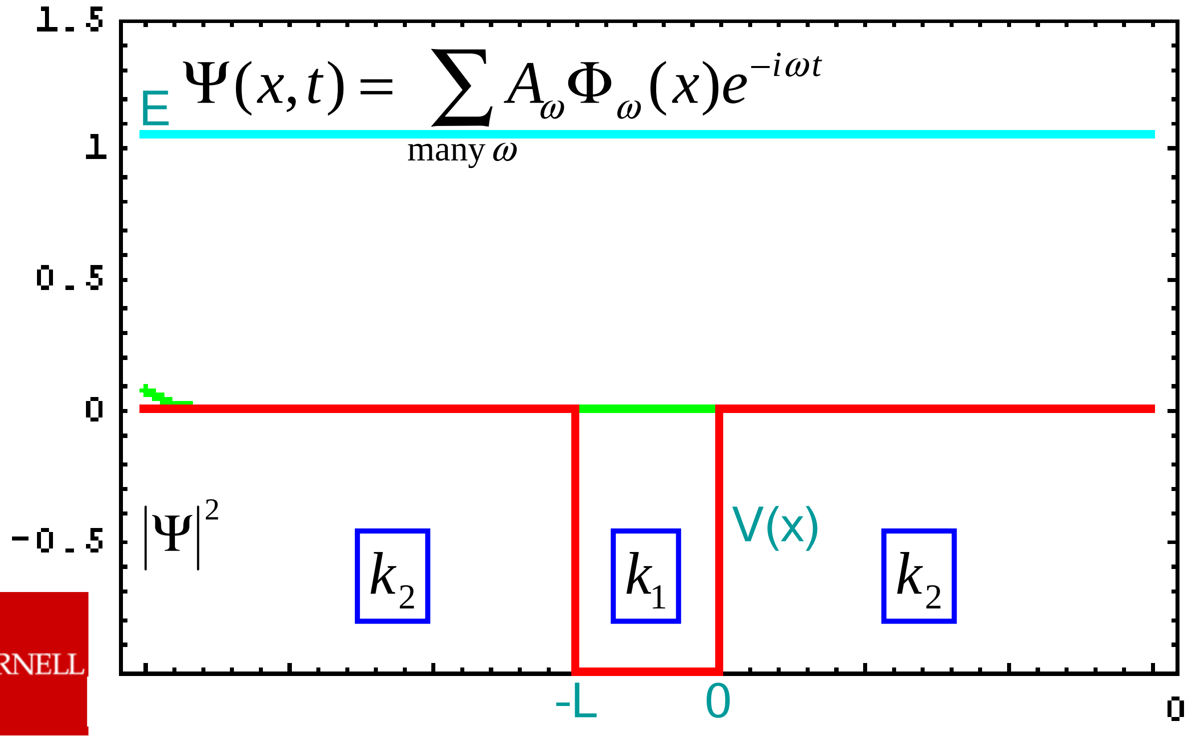
Scattering by a one-dimensional well

04/08/2005

$$\Phi_\omega(x)e^{-i\omega t}=\begin{cases}\tilde{A}e^{ik_2x}+\tilde{B}e^{-ik_2x} & \text{for } x<-L\\ C\frac{k_1+k_2}{2k_1}e^{ik_1x}+C\frac{k_1-k_2}{2k_1}e^{-ik_1x} & \text{for } -L\le x<0\\ Ce^{ik_2x} & \text{for } x\ge 0\end{cases}$$

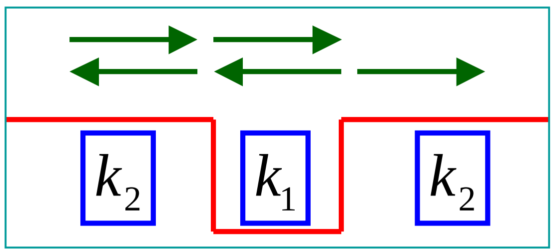
$$\left.\begin{aligned}k_2\frac{\tilde{A}}{C}e^{-ik_2L}+k_2\frac{\tilde{B}}{C}e^{ik_2L}&=k_2\left(\frac{k_1+k_2}{2k_1}e^{-ik_1L}+\frac{k_1-k_2}{2k_1}e^{ik_1L}\right)\\ k_2\frac{\tilde{A}}{C}e^{-ik_2L}-k_2\frac{\tilde{B}}{C}e^{ik_2L}&=k_1\left(\frac{k_1+k_2}{2k_1}e^{-ik_1L}-\frac{k_1-k_2}{2k_1}e^{ik_1L}\right)\end{aligned}\right\}\begin{aligned}\frac{\tilde{A}}{C}&=e^{ik_2L}\frac{(k_2+k_1)^2e^{-ik_1L}-(k_1-k_2)^2e^{ik_1L}}{4k_2k_1}\\ \frac{\tilde{B}}{C}&=e^{-ik_2L}\frac{(k_2-k_1)(k_1+k_2)(e^{-ik_1L}-e^{ik_1L})}{4k_2k_1}\end{aligned}$$

$$\Psi(x,t)=\sum_{\text{many }\omega}A_\omega\Phi_\omega(x)e^{-i\omega t}$$



Example: Electron scattering from a positive charge.

Considered waves:

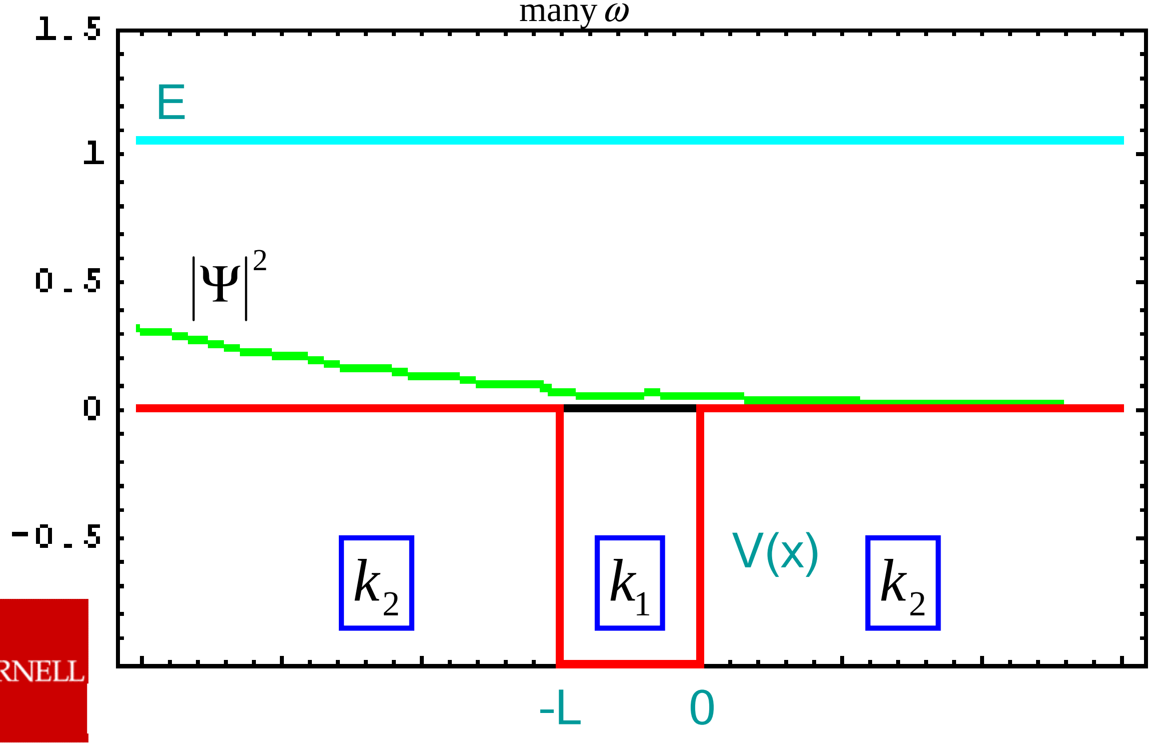


Georg.Hoffstaetter@Cornell.edu

Reduced momentum spread

The scattering of an infinite mono energetic wave is resembled more closely when the wave packet has less energy spread and is therefore longer:

$$\Psi(x,t) = \sum_{\text{many } \omega} A_\omega \Phi_\omega(x) e^{-i\omega t}$$



Different reflection and transmission for different energies is also used for light waves in thin coatings of lenses. Such lenses have less attenuation than those bases on selective absorption.

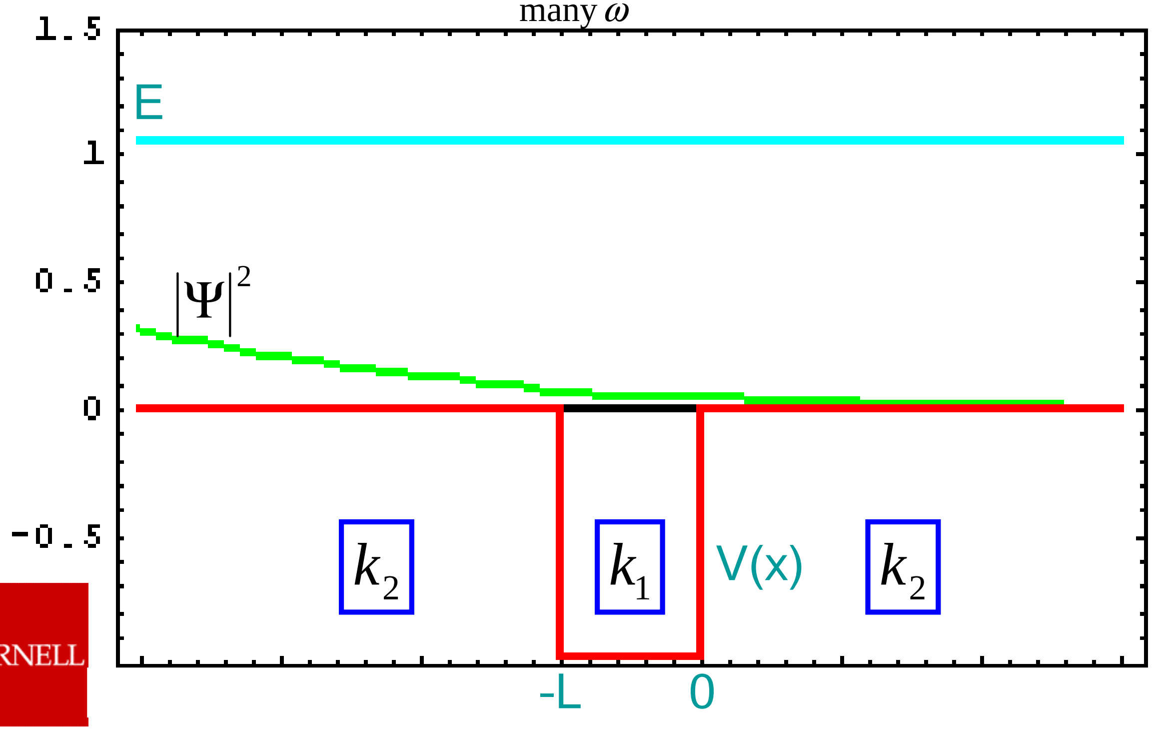
Georg.Hoffstaetter@Cornell.edu

04/08/2005

The Ramsauer Effect

$$\left.\begin{aligned}\frac{\tilde{A}}{C} &= e^{ik_2L}\,\frac{(k_2+k_1)^2 e^{-ik_1L}-(k_1-k_2)^2 e^{ik_1L}}{4k_2k_1}\\ \frac{\tilde{B}}{C} &= e^{-ik_2L}\,\frac{(k_2-k_1)(k_1+k_2)(e^{-ik_1L}-e^{ik_1L})}{4k_2k_1}\end{aligned}\right\}\rightarrow\left\{\begin{aligned}\frac{\tilde{A}}{C} &= e^{ik_2L}\,\frac{2k_1k_2\cos(k_1L)-i(k_1^2+k_2^2)\sin(k_1L)}{2k_2k_1}\\ \frac{\tilde{B}}{C} &= ie^{-ik_2L}\,\frac{k_1^2-k_2^2}{2k_2k_1}\sin(k_1L)\end{aligned}\right.$$

$$\Psi(x,t)=\sum_{\text{many }\omega} A_\omega \Phi_\omega(x) e^{-i\omega t}$$



Whenever the energy is such that $\sin(k_1L)=0$, no reflection occurs.

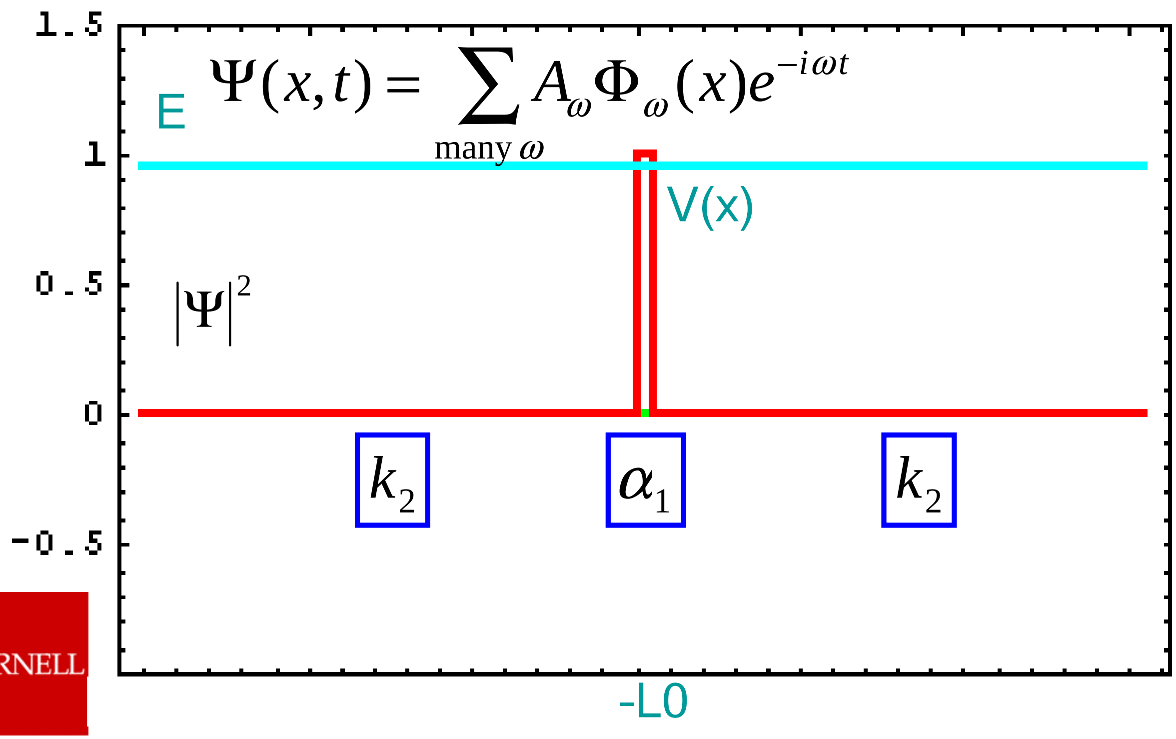
This is the same condition as for the energy of a bound state in an infinite potential well.

Georg.Hoffstaetter@Cornell.edu

Barrier penetration: Tunneling

$$\xrightarrow[\sin(i\alpha_1)=i\sinh(\alpha_1)]{\cos(i\alpha_1)=\cosh(\alpha_1)}$$

$$\left.\begin{aligned}\frac{\tilde{A}}{C} &= e^{ik_2L}\,\frac{2k_1k_2\cos(k_1L)-i(k_1^2+k_2^2)\sin(k_1L)}{2k_2k_1}\\ \frac{\tilde{B}}{C} &= ie^{-ik_2L}\,\frac{k_1^2-k_2^2}{2k_2k_1}\sin(k_1L)\end{aligned}\right\}\rightarrow\left\{\begin{aligned}\frac{\tilde{A}}{C} &= e^{ik_2L}\,\frac{2\alpha_1k_2\cosh(\alpha_1L)+i(\alpha_1^2-k_2^2)\sinh(\alpha_1L)}{2k_2\alpha_1}\\ \frac{\tilde{B}}{C} &= -ie^{-ik_2L}\,\frac{\alpha_1^2+k_2^2}{2k_2\alpha_1}\sinh(\alpha_1L)\end{aligned}\right.$$



|B|>0: No reflection free transport is possible.

Georg.Hoffstaetter@Cornell.edu