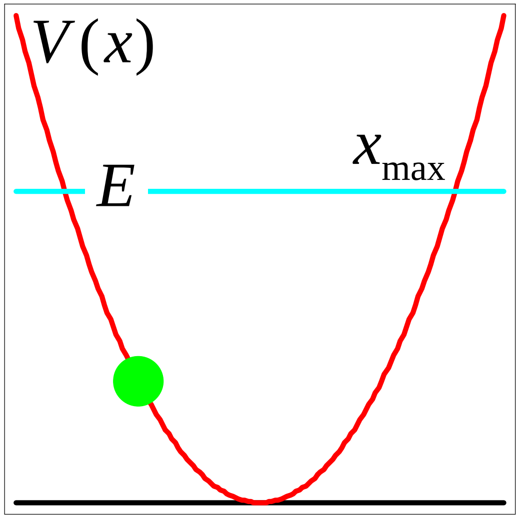
The harmonic oscillator:

The **classical oscillator** often occurs in nature and is often a good approximation for small oscillations.

$$V(x)=\tfrac{1}{2}Cx^2 \quad\rightarrow\quad m\ddot{x}=-Cx \quad\rightarrow\quad \text{classical oscillation with } \omega_0=\sqrt{\tfrac{C}{m}}$$

$$\text{Maximum oscillation amplitude}: x_{\max}^2=\frac{2E}{m\omega_0^2}$$



$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Phi(x)+\tfrac{1}{2}Cx^2\,\Phi(x)=E\,\Phi(x)$$

$$\frac{\hbar\omega_0}{2}\{-\frac{\hbar}{m\omega_0}\frac{\partial^2}{\partial x^2}+\frac{m\omega_0}{\hbar}x^2\}\Phi(x)=E\,\Phi(x)$$

Simplification: $a=\sqrt{\frac{\hbar}{m\omega_0}}\,,\quad \xi={}^{x}\!/\!_{a}$

$$\frac{\hbar\omega_0}{2}\{-\frac{\partial^2}{\partial \xi^2}+\xi^2\}\Phi(x)=E\,\Phi(x)$$

$$\Phi_n(x)=A_n\,f_n(\xi)\,e^{-\frac{1}{2}\xi^2} \quad\Rightarrow\quad \frac{\hbar\omega_0}{2}\{-\frac{\partial^2}{\partial\xi^2}+2\xi\frac{\partial}{\partial\xi}+1\}\,f_n(\xi)=E_n\;f_n(\xi)$$

$$f_0(\xi)=1\,,\quad \Phi_0(x)=Ae^{-\frac{1}{2}\xi^2}\,,\quad E_0=\frac{\hbar\omega_0}{2}$$

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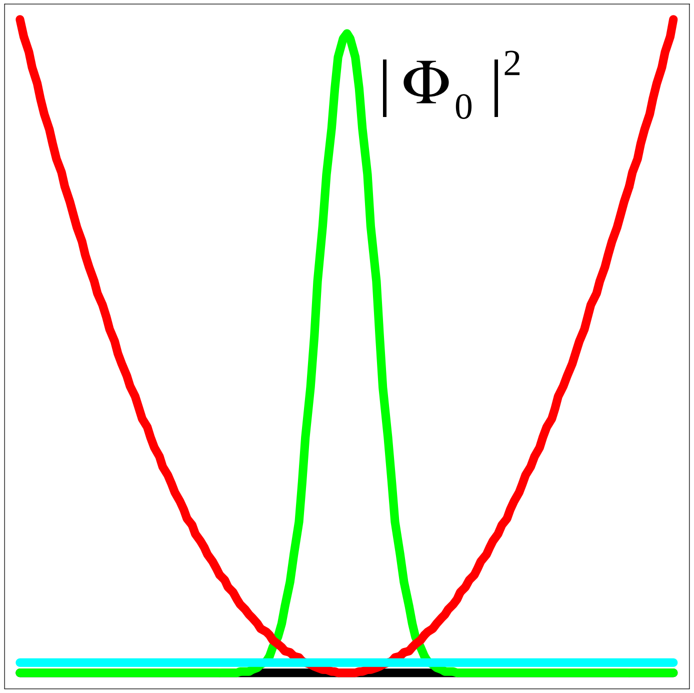
Ground states

02/28/2005

Gaussian function:

$$f_0(\xi) = 1, \quad \Phi_0(x) = A e^{-\frac{1}{2}\xi^2}, \quad E_0 = \frac{\hbar\omega_0}{2}$$

Carl Friedrich Gauss
(1777-1855, Germany)



The falloff for large **x** is even faster than for the finite potential well where it was $e^{-\sqrt{\frac{2m}{\hbar}}\sqrt{V_0 - E}\,x}$

This is due to the fact that now the potential increases with **x**: $V(x) \propto x^2$

All wave functions are dominated by the same form at large **x**.

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Exited states:

$$a = \sqrt{\frac{\hbar}{m\omega_0}}\,, \quad \xi = x/a$$

$$\frac{\hbar\omega_0}{2}\{-\frac{\partial^2}{\partial\xi^2}+\xi^2\}\Phi(x) = E\,\Phi(x)$$

$$\Phi_n(x) = A_n f_n(\xi) e^{-\frac{1}{2}\xi^2} \quad \Rightarrow \quad \frac{\hbar\omega_0}{2}\{-\frac{\partial^2}{\partial\xi^2}+2\xi\frac{\partial}{\partial\xi}+1\} f_n(\xi) = E_n\ f_n(\xi)$$

$$f_n(\xi) = \sum_{j=0}^{\infty} c_j \xi^j \quad \Rightarrow \quad \sum_{j=0}^{\infty}\{-c_{j+2}\frac{(j+2)(j+1)}{2}+c_j(j+\tfrac{1}{2})\}\xi^j = \frac{E_n}{\hbar\omega_0} f_n(\xi)$$

$$f_n(\xi) = \sum_{j=0}^{\infty} c_j \xi^j \quad \Rightarrow \quad c_{j+2}\frac{(j+2)(j+1)}{2} = c_j(j+\tfrac{1}{2}-\tfrac{E_n}{\hbar\omega_0})$$

The series terminates when $\frac{E_n}{\hbar\omega_0}$ equals $n+\frac{1}{2}$ for some integer n, yielding an nth order polynomial $\mathbf{f_n}$.

This leads to a wave function $\Phi_n(x) = A_n f_n(\xi) e^{-\frac{1}{2}\xi^2}$ with n nodes.

Therefore this leads to **all possible wave functions.**

$$E_n = \hbar\omega_0(n+\tfrac{1}{2})$$

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Hermit polynomials

02/28/2005

Stationary states of the harmonic oscillator:

$$\Phi_n(x) = \frac{A}{\sqrt{n!}} \frac{1}{\sqrt{2^n}} e^{-\frac{1}{2}\xi^2} H_n(\xi)$$

Hermite polynomials: $H_n(\xi) = (-1)^n e^{\xi^2} \frac{\partial^n}{\partial \xi^n} e^{-\xi^2}$

Normalization: $\int_{-\infty}^{\infty} |\Phi(x)|^2 \, dx = 1$ **leads to the constant A.**

Charles Hermite

1822-1901

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Lowest Eigenvalue :

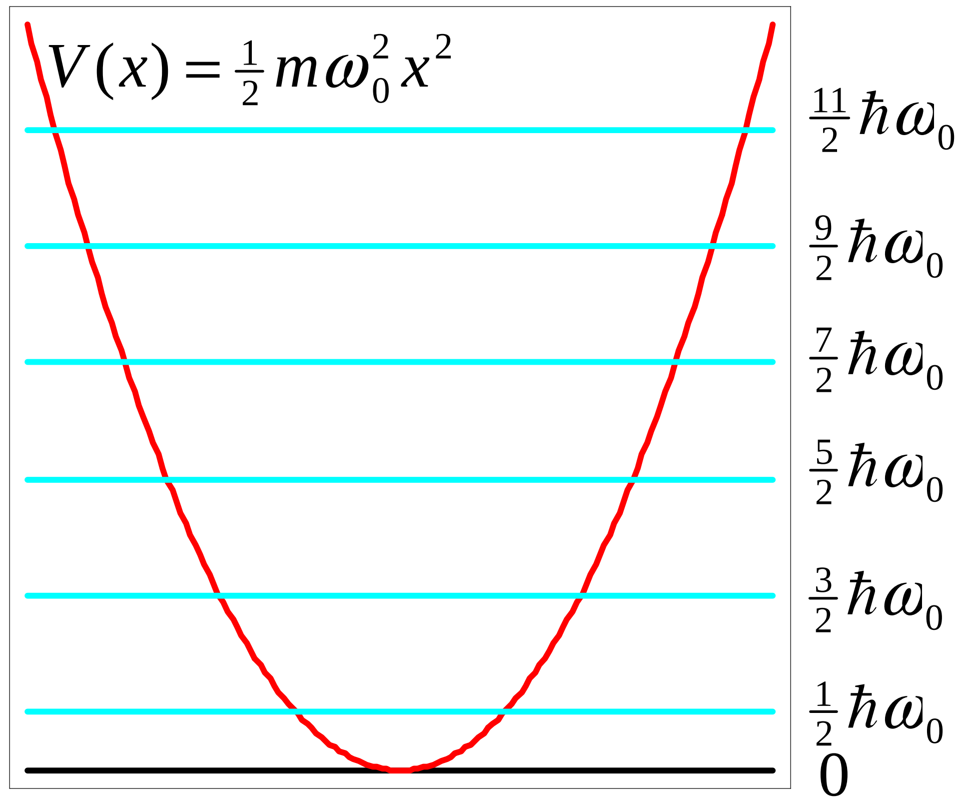
The chain of eigenfunctions Φ_n
where **n is a positive integer**.

The Schrödinger equation of
an harmonic oscillator
has eigenvalues

$$E_n = \hbar\omega_0 (n + \tfrac{1}{2})$$

With the lowest possible energy or
ground state energy

$$E_0 = \tfrac{1}{2}\hbar\omega_0$$



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Probability amplitudes for eigenstates

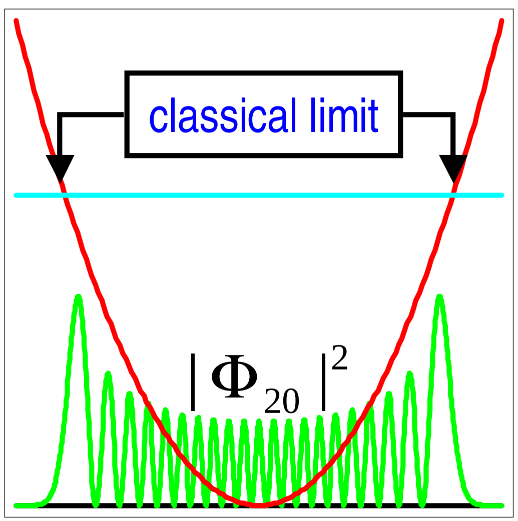
$$\Phi_n = \frac{A}{\sqrt{n!}} \frac{1}{\sqrt{2^n}} e^{-\frac{1}{2}\xi^2} H_n(\xi), \quad \xi = x/a, \quad a = \sqrt{\frac{\hbar}{m\omega_0}}, \quad E_n = \hbar\omega_0 (n + \tfrac{1}{2})$$

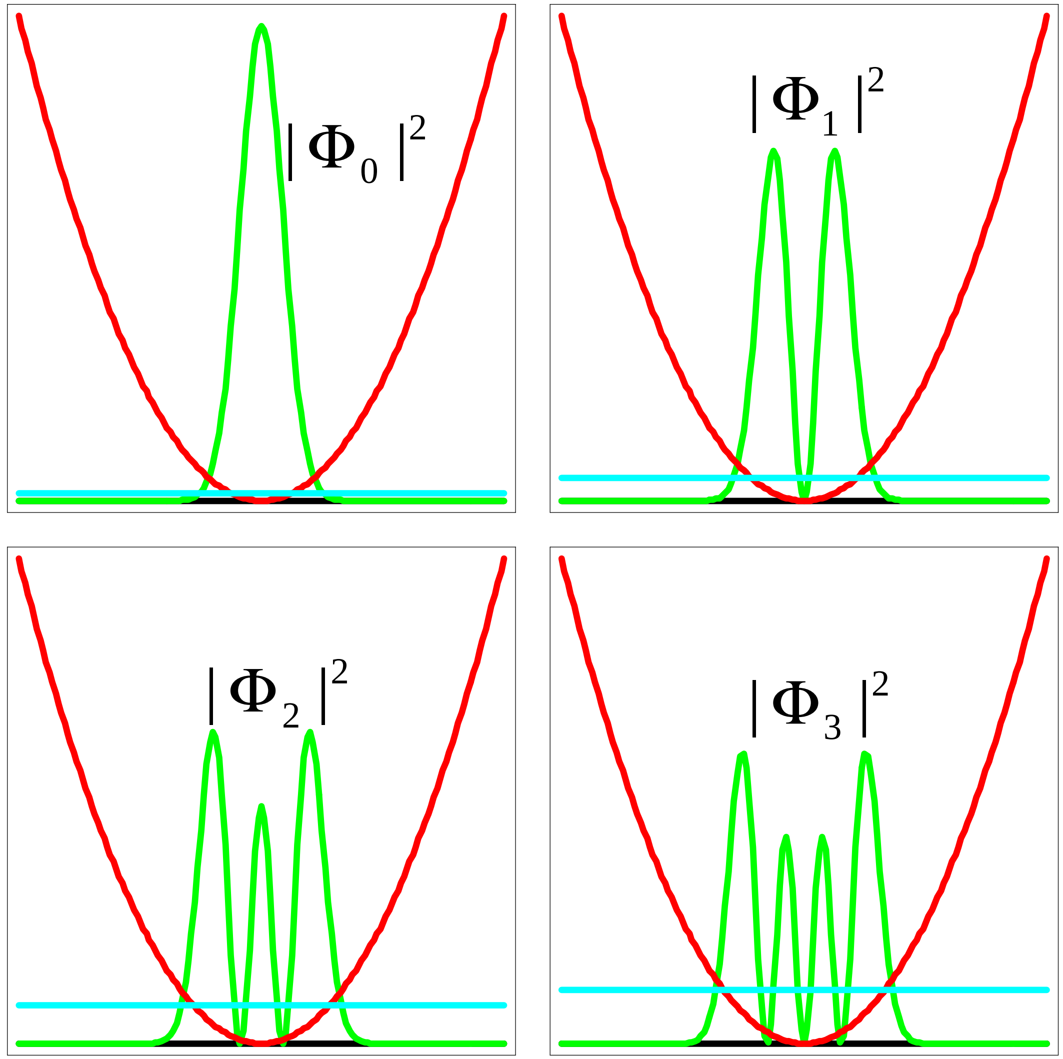
$$H_0(\xi) = 1$$

$$H_1(\xi) = 2\xi$$

$$H_2(\xi) = 4\xi^2 - 2$$

$$H_3(\xi) = 8\xi^3 - 12\xi$$





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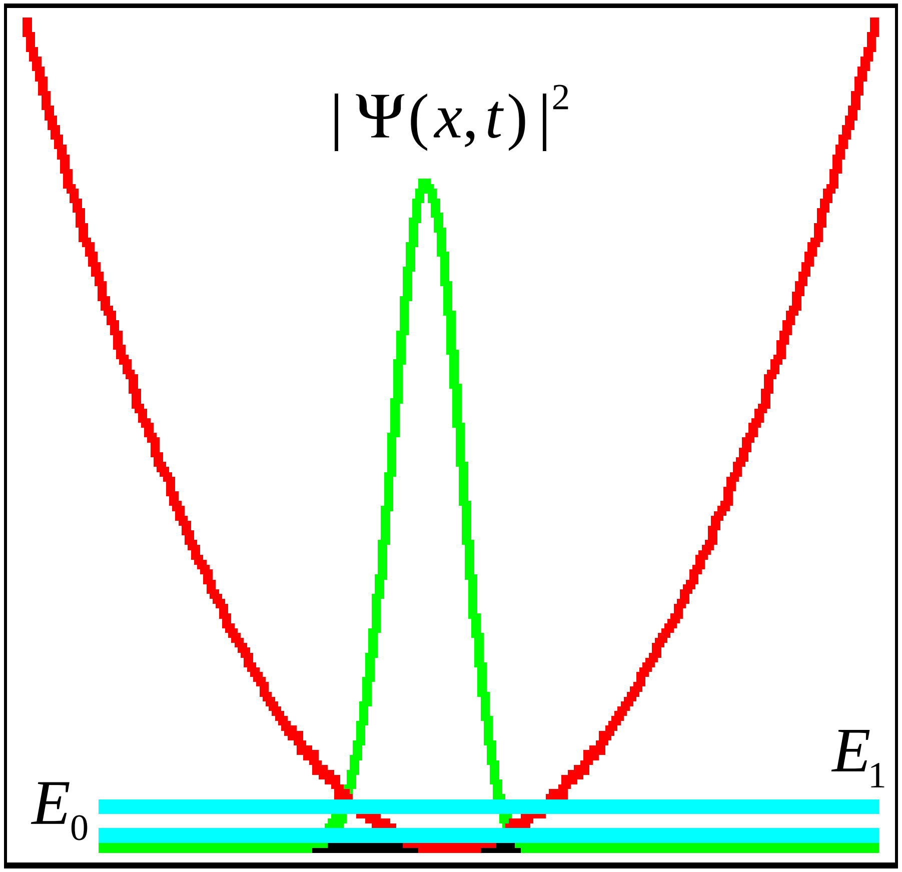
Time dependent states in the square potential

$$\Psi(x,t) \propto \Phi_0(x)e^{-i\frac{E_0}{\hbar}t} + \tfrac{1}{2}\Phi_1(x)e^{-i\frac{E_1}{\hbar}t}$$



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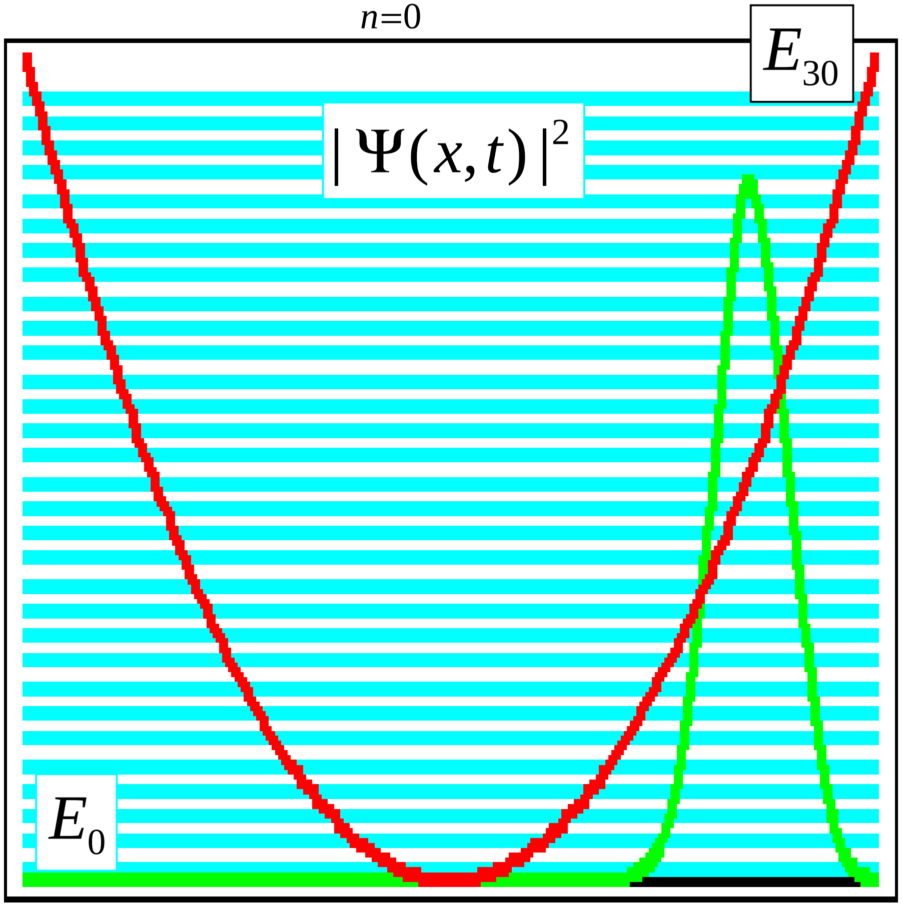
Time dependent states in the square potential

$$\Psi(x,t) \propto \Phi_0(x)e^{-i\frac{E_0}{\hbar}t} + \tfrac{1}{2}\Phi_1(x)e^{-i\frac{E_1}{\hbar}t}$$

$$\Psi(x,t) = \sum_{n=0}^{30} A_n \Phi_n(x) e^{-i\frac{E_n}{\hbar}t}$$



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