3) The wave-particle duality and bound states

The wave character of light and the photon character of light are both but different manifestations of the same thing.

Similarly the particle character of electrons, atoms, and neutrons and their wave character are both but different manifestations of the same thing.

Free particles: $p=\hbar k\,,\quad E=\hbar\omega$

Particles in a potential: $E=\frac{p^2}{2m_0}+V \rightarrow \omega(k)=\frac{1}{\hbar}[\frac{(\hbar k)^2}{2m_0}+V]$

(non-relativistic)

To describe waves more generally, one needs a wave equation for this dispersion relation ω**(k).**

Erwin Schrödinger
(1887-1961, Austria)
1933 Nobel prize

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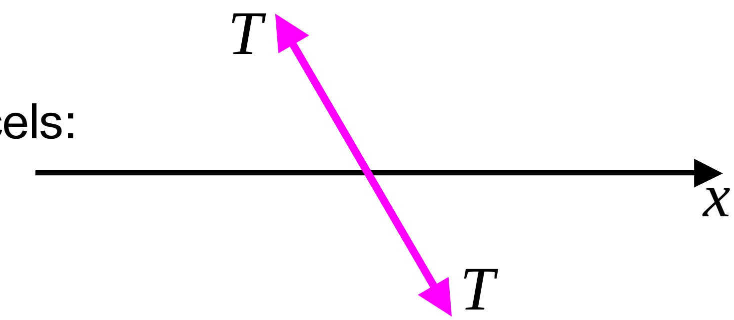
A wave equation for an oscillating string

A string with uniform mass per length supports oscillations of the form $y(x,t) = A\sin(k[x - vt])$

If the mass distribution is not uniform, a more detailed analysis is needed:

No curvature of string:

no acceleration since the tension T cancels:

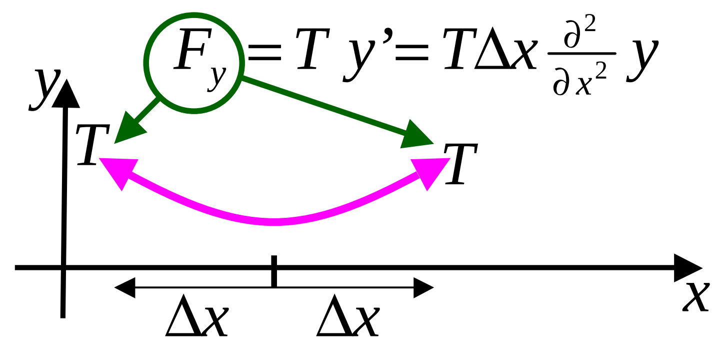


$$2T\Delta x \frac{\partial^2}{\partial x^2} y = \Delta m \frac{\partial^2}{\partial t^2} y$$

$$\Delta m = 2\Delta x \Lambda$$

$$\frac{T}{\Lambda} \frac{\partial^2}{\partial x^2} y = \frac{\partial^2}{\partial t^2} y$$

$$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} y(x,t)\,; \quad v^2 = \frac{T}{\Lambda}$$



Initial value problem: $y(x,t_0) = y_0(x)$

What is $y(x,t)$

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Initial value problem for a guitar string

02/14/2005

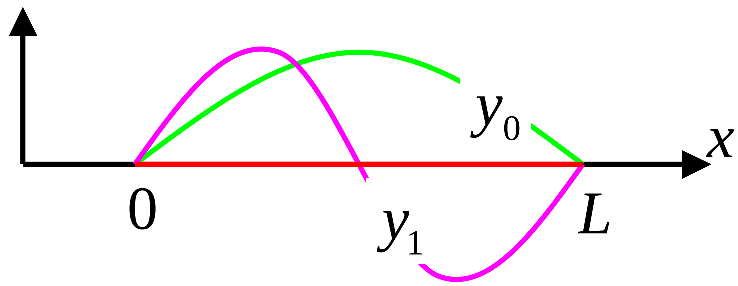
Assuming uniform mass distribution: v=const.

$$\frac{\partial^2}{\partial x^2} y(x,t) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} y(x,t)$$

For oscillations with frequency ω: $y_\omega(x)e^{-i\omega t}$

$$\frac{\partial^2}{\partial x^2} y_\omega(x) = -\frac{\omega^2}{v^2} y_\omega(x) \quad \Rightarrow \quad y_\omega = \sin(kx)$$

$$y_\omega(0) = 0\,; \quad y_\omega(L) = 0 \quad \Rightarrow \quad k = k_n = \tfrac{\pi}{L} n$$



Not all frequencies are possible, but only $\omega = \omega_n = v k_n = v \frac{\pi}{L} n$

$$y_n(x,t) = A_n \sin(k_n x) e^{-i\omega_n t}$$

Linearity of the wave equation: If y_1 and y_2 are both solutions, so is y_1+y_2

General solution: $$y(x,t) = \sum_{n=1}^{\infty} A_n \sin(k_n x) e^{-i\omega_n t}$$

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The A_n are determined by $y(x,0) = \sum_{n=1}^{\infty} A_n \sin(k_n x) = y_0(x)$

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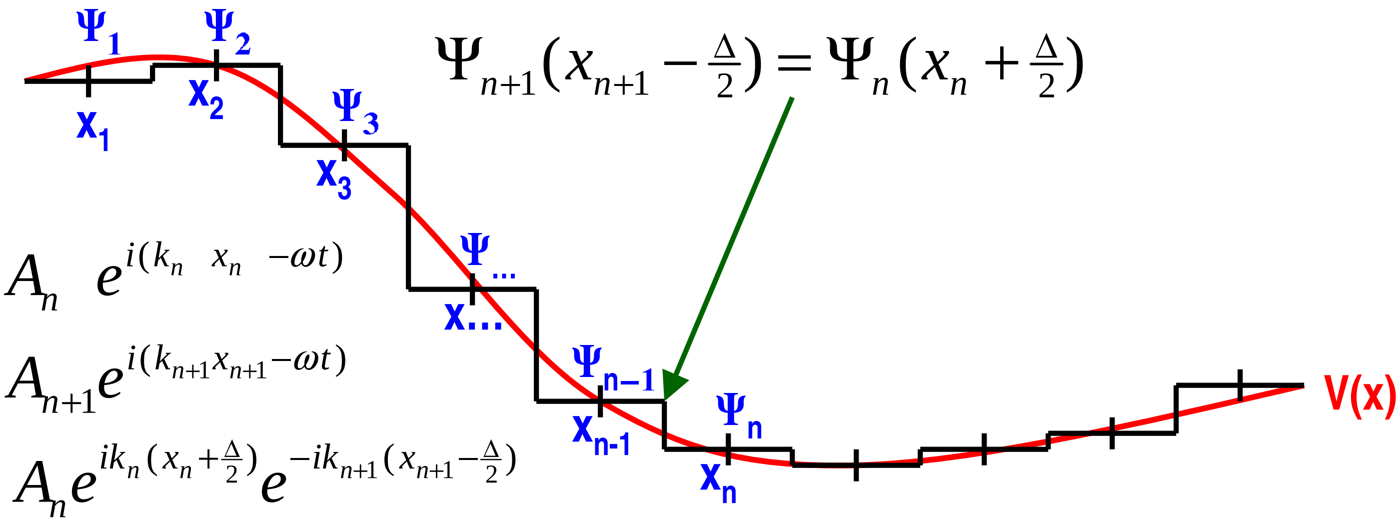
The stationary Schrödinger equation

The wave number **k** changes with position **x** according to: $\frac{\hbar^2 k^2}{2m} + V(x) = E$

Particle with fixed constant energy **E:** ω does not change

For regions where **V** does not change with **x**, a wave with **k** and ω is given by $\Psi = A e^{i(kx - \omega t)}$



$$\Psi_{n+1}(x_{n+1} - \tfrac{\Delta}{2}) = \Psi_n(x_n + \tfrac{\Delta}{2})$$

$$\Psi_n = A_n e^{i(k_n x_n - \omega t)}$$

$$\Psi_{n+1} = A_{n+1} e^{i(k_{n+1} x_{n+1} - \omega t)}$$

$$A_{n+1} = A_n e^{i k_n (x_n + \frac{\Delta}{2})} e^{-i k_{n+1} (x_{n+1} - \frac{\Delta}{2})}$$

$$\Psi_{n+1} = A_n e^{i[k_n (x_n + \frac{\Delta}{2}) - k_{n+1}(x_{n+1} - \frac{\Delta}{2}) + k_{n+1} x_{n+1}]} e^{-i\omega t} = \Psi_n e^{i[k_n \frac{\Delta}{2} + k_{n+1} \frac{\Delta}{2}]}$$

$$= \Psi_n e^{i \frac{k_{n+1} + k_n}{2} \Delta} \approx \Psi_n + i k_n \Psi_n \Delta \quad \rightarrow \quad \frac{\partial}{\partial x} \Psi(x,t) = i k \Psi$$

Conclusion: whenever $k\,\Psi$ needs to be computed, one can use $-i \frac{\partial}{\partial x} \Psi$

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$$\Psi(x,t) = \Phi(x) e^{-i\omega t} \quad \rightarrow \quad -\frac{\hbar^2}{2m} \frac{\partial}{\partial x} \frac{\partial}{\partial x} \Phi(x) + V(x)\Phi(x) = E\,\Phi(x)$$

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