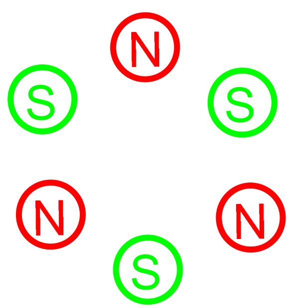
Sextupoles (revisited)

CHESS & LEPP

$$\psi = \Psi_3 \,\mathrm{Im}\{(x - iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

C_3 Symmetry



i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = 3!\Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

Chromaticity and its Correction

Chromaticity ξ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial \upsilon}{\partial \delta}\delta + \dots$$

$$\xi = \tfrac{\partial \upsilon}{\partial \delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity ξ_0 = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s} \qquad \xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_x k_2)d\hat{s}$$

Typically the the chormaticity ξ is chosen to be slightly positive, between 0 and 3.

Perturbations

$$\begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -K & 0 \end{pmatrix} \begin{pmatrix} x \\ a \end{pmatrix} + \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} = \sqrt{2J}\, \underline{\beta}\, \vec{S}$$

This would be a solution with constant J and ϕ when $\Delta f = 0$.

Variation of constants:

$$\frac{J'}{\sqrt{2J}} \underline{\beta}\, \vec{S} + \sqrt{2J}\ \phi_0' \begin{pmatrix} 0 & \sqrt{\beta} \\ -\frac{1}{\sqrt{\beta}} & -\frac{\alpha}{\sqrt{\beta}} \end{pmatrix} \vec{S} = \begin{pmatrix} 0 \\ \Delta f \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}} \vec{S} + \sqrt{2J}\ \phi_0' \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{S} = \underline{\beta}^{-1} \begin{pmatrix} 0 \\ \Delta f \end{pmatrix} \quad \text{with} \quad \underline{\beta}^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

$$\frac{J'}{\sqrt{2J}} = \cos(\psi + \phi_0)\sqrt{\beta}\Delta f \quad , \quad \sqrt{2J}\ \phi_0' = -\sin(\psi + \phi_0)\sqrt{\beta}\Delta f$$

Simplification of linear motion

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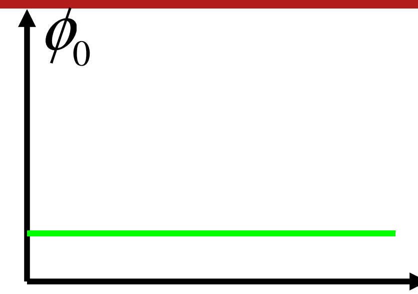
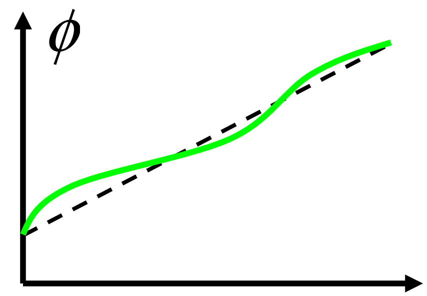
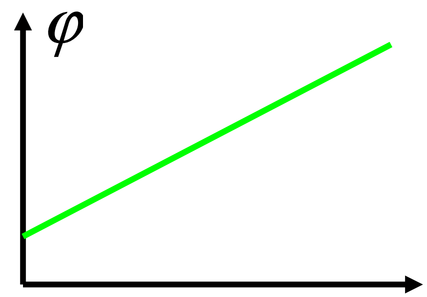
$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi + \phi_0) \\ \cos(\psi + \phi_0) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi_0' = 0 \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi \\ \cos\phi \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \phi' = \frac{1}{\beta} \end{matrix}$$



$$\begin{pmatrix} x \\ a \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi - \mu\frac{s}{L} + \varphi) \\ \cos(\psi - \mu\frac{s}{L} + \varphi) \end{pmatrix} \Rightarrow \begin{matrix} J' = 0 \\ \varphi' = \mu\frac{1}{L} \end{matrix}$$



$$\tilde{\psi} = \psi - \mu\frac{s}{L} \Rightarrow \tilde{\psi}(s+L) = \tilde{\psi}(s)$$

Corresponds to Floquet's Theorem

Quasi-periodic Perturbation

$$J' = \cos(\psi + \phi_0)\sqrt{2J\beta}\Delta f \quad , \quad \phi_0' = -\sin(\psi + \phi_0)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

$$J' = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f \quad , \quad \varphi' = \mu\tfrac{1}{L} - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f$$

New independent variable $\vartheta = 2\pi \frac{s}{L}$

$$\tfrac{d}{d\vartheta} J = \cos(\tilde{\psi} + \varphi)\sqrt{2J\beta}\Delta f \, \tfrac{L}{2\pi} \quad , \quad \tfrac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi} + \varphi)\sqrt{\tfrac{\beta}{2J}}\Delta f \, \tfrac{L}{2\pi}$$

$$\Delta f(x) = \Delta f(\sqrt{2J\beta}\sin(\tilde{\psi} + \varphi))$$

The perturbations are 2π periodic in ϑ and in φ

φ is approximately $\varphi \approx \upsilon \cdot \vartheta$

For irrational ν, the perturbations are quasi-periodic.

Tune Shift with Amplitude

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$$\frac{d}{d\vartheta}J=\cos(\tilde{\Psi}+\varphi)\sqrt{2J\beta}\Delta f\,\frac{L}{2\pi}\quad,\quad \frac{d}{d\vartheta}\varphi=\upsilon-\sin(\tilde{\Psi}+\varphi)\sqrt{\frac{\beta}{2J}}\Delta f\,\frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi=\partial_J H\quad,\quad \frac{d}{d\vartheta}J=-\partial_\phi H\quad,\quad H(\varphi,J,\vartheta)=\upsilon\cdot J-\frac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}$$

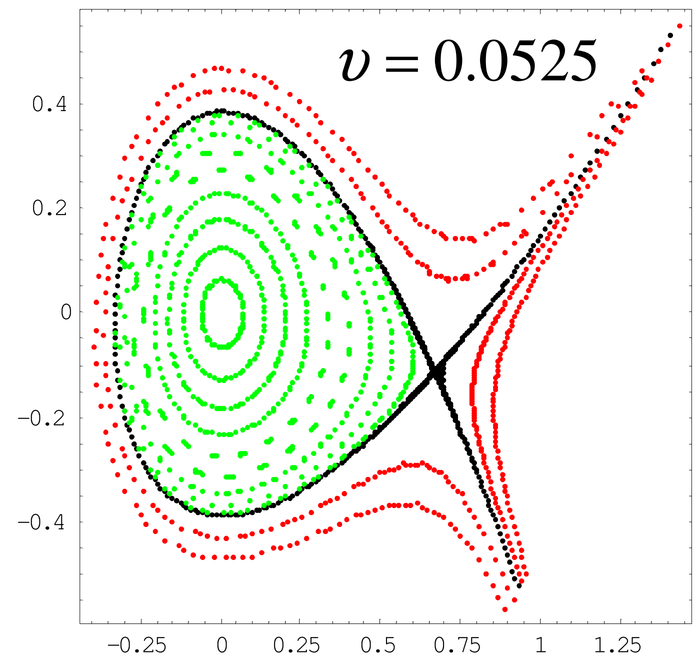
The motion remains Hamiltonian in the perturbed coordinates !

If there is a part in $\partial_J H$ that does not depend on φ, s $\Rightarrow$ Tune shift

The effect of other terms tends to average out.

$$\varphi(\vartheta)-\varphi_0\approx\vartheta\cdot\partial_J\langle H\rangle_{\varphi,\vartheta}(J)$$

$$\upsilon(J)=\upsilon+\partial_J\langle\Delta H\rangle_{\varphi,\vartheta}(J)$$



Tune Shift Examples

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$$H(\varphi,J)=\upsilon\cdot J-\tfrac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}\quad,\quad \Delta\upsilon(J)=\partial_J\langle\Delta H\rangle_{\varphi,\vartheta}$$

Quadrupole: $\Delta f=-\Delta k\,x$

$$\Delta H=\tfrac{L}{2\pi}\Delta k\tfrac{1}{2}x^2=\tfrac{L}{2\pi}\Delta k\,J\beta\sin^2(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{1}{2\pi}\int_0^{2\pi}\Delta k\beta\,d\vartheta\,L\frac{J}{4\pi}=\int_0^L\Delta k\beta\,ds\frac{J}{4\pi}\Rightarrow\Delta\upsilon=\tfrac{1}{4\pi}\oint\Delta k\beta\,ds$$

Sextupole: $\Delta f=-k_2\tfrac{1}{2}x^2$

$$\Delta H=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}x^3=\tfrac{L}{2\pi}k_2\tfrac{1}{3!}\sqrt{2J\beta}^{\,3}\sin^3(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=0\quad\Rightarrow\quad\Delta\upsilon=0$$

Octupole: $\Delta f=-k_3\tfrac{1}{3!}x^3$

$$\Delta H=\tfrac{L}{2\pi}k_3\tfrac{1}{4!}x^4=\tfrac{L}{2\pi}k_3\tfrac{1}{3!}(J\beta)^2\sin^4(\tilde{\psi}+\varphi)$$

$$\langle\Delta H\rangle_{\varphi,\vartheta}=\tfrac{J^2}{3!2\pi}\oint k_3\beta^2 ds\left\langle\tfrac{1}{2^4}(e^{i\varphi}-e^{-i\varphi})^4\right\rangle_\varphi\Rightarrow\Delta\upsilon=J\tfrac{1}{16\pi}\oint k_3\beta^2 ds$$

Nonlinear Resonances

$$\frac{d}{d\vartheta}J = \cos(\tilde{\psi}+\varphi)\sqrt{2J\beta}\Delta f\,\frac{L}{2\pi}\quad,\quad \frac{d}{d\vartheta}\varphi = \upsilon - \sin(\tilde{\psi}+\varphi)\sqrt{\frac{\beta}{2J}}\Delta f\,\frac{L}{2\pi}$$

$$\frac{d}{d\vartheta}\varphi = \partial_J H\quad,\quad \frac{d}{d\vartheta}J = -\partial_\phi H\quad,\quad H(\varphi,J,\vartheta) = \upsilon\cdot J - \frac{L}{2\pi}\int_0^x \Delta f(\hat{x},s)\,d\hat{x}$$

The effect of the perturbation is especially strong when

$\cos(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$ or $\sin(\tilde{\psi}+\varphi)\sqrt{\beta}\Delta f$

has contributions that hardly change, i.e. the change of

$\sqrt{\beta(\vartheta)}\Delta f(x(\vartheta),\vartheta)$ is in resonance with the rotation angle $\varphi(\vartheta)$.

Periodicity allows Fourier expansion:

$$H(\varphi,J,\vartheta) = \sum_{n,m=-\infty}^{\infty}\widehat{H}_{nm}(J)\,e^{i[n\vartheta+m\varphi]} = \sum_{n,m=-\infty}^{\infty}H_{nm}(J)\cos(n\vartheta+m\varphi+\Psi_{nm}(J))$$

$$H_{00}(J) = \langle H(\varphi,J,s)\rangle_{\varphi,s} \Rightarrow$$ Tune shift