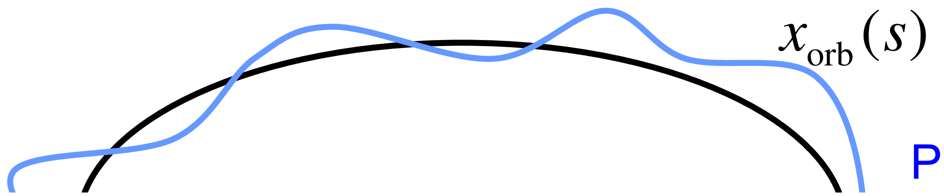
Oscillations around a distorted Orbit



Particles oscillate around this periodic orbit, not around the design orbit.

$$\vec{z} = \vec{z}_\beta + \vec{z}_{\mathrm{orb}}$$

$$\vec{z}_{\mathrm{orb}}(s) = \underline{M}\,\vec{z}_{\mathrm{orb}}(0) + \Delta\vec{z}(s)$$

$$\vec{z}_\beta(s) + \vec{z}_{\mathrm{orb}}(s) = \vec{z}(s) = \underline{M}\,\vec{z}(0) + \Delta\vec{z}(s) = \underline{M}[\vec{z}_\beta(0) + \vec{z}_{\mathrm{orb}}(0)] + \Delta\vec{z}(s)$$

$$= \underline{M}\,\vec{z}_\beta(0) + \vec{z}_{\mathrm{orb}}(s)$$

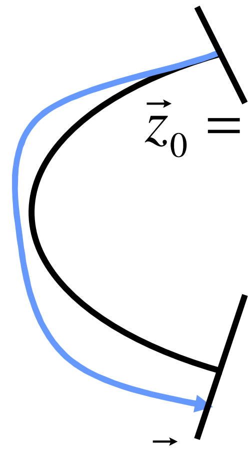
$$\boxed{\vec{z}_\beta(L) = \underline{M}_0\vec{z}_\beta(0)}$$

The distorted orbit does not change the linear transport matrix.

Dispersion Integral

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$



$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s}) \begin{pmatrix} 0 \\ \delta\kappa(\hat{s}) \end{pmatrix} d\hat{s}$$

$$\Rightarrow \quad \vec{D}(s) = \int_0^s \underline{M}(s-\hat{s}) \begin{pmatrix} 0 \\ \kappa(\hat{s}) \end{pmatrix} ds'$$

$$\Delta\kappa = \delta\kappa$$

$$D(s) = \sqrt{\beta(s)} \int_0^s \kappa(\hat{s}) \sqrt{\beta(\hat{s})} \sin(\psi(s) - \psi(\hat{s})) d\hat{s}$$

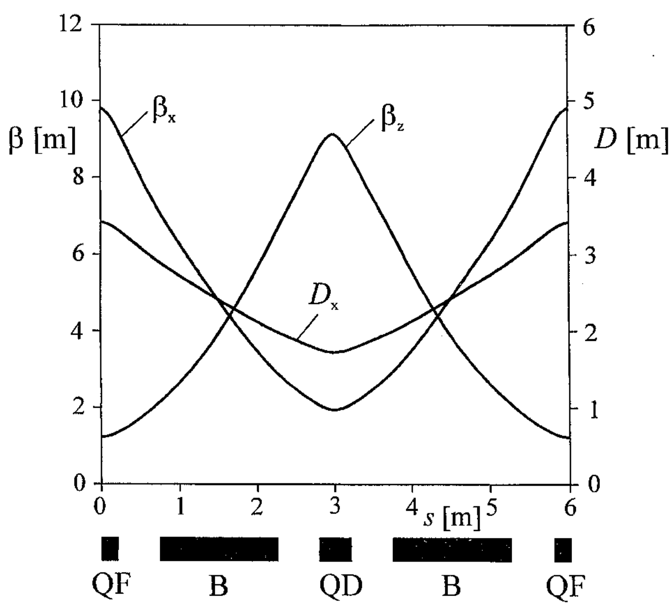
The FODO Cell

Alternating gradients allow focusing in both transverse plains. Therefore focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.

$$L_{FoDo} \approx 6\text{m}\,, \quad \varphi \approx 22.5°\,, \quad \mu_{FoDo} \approx \tfrac{\pi}{2}$$

$$\overline{\beta} \approx 3.8\text{m}$$

$$\beta_{\max} \approx 10.2\text{m}\,, \quad \beta_{\min} \approx 1.8\text{m}$$



$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for an accelerator section that consists of many FODO cells. Often large sections of an accelerator consist of FODOs.

The Low Beta Insertion

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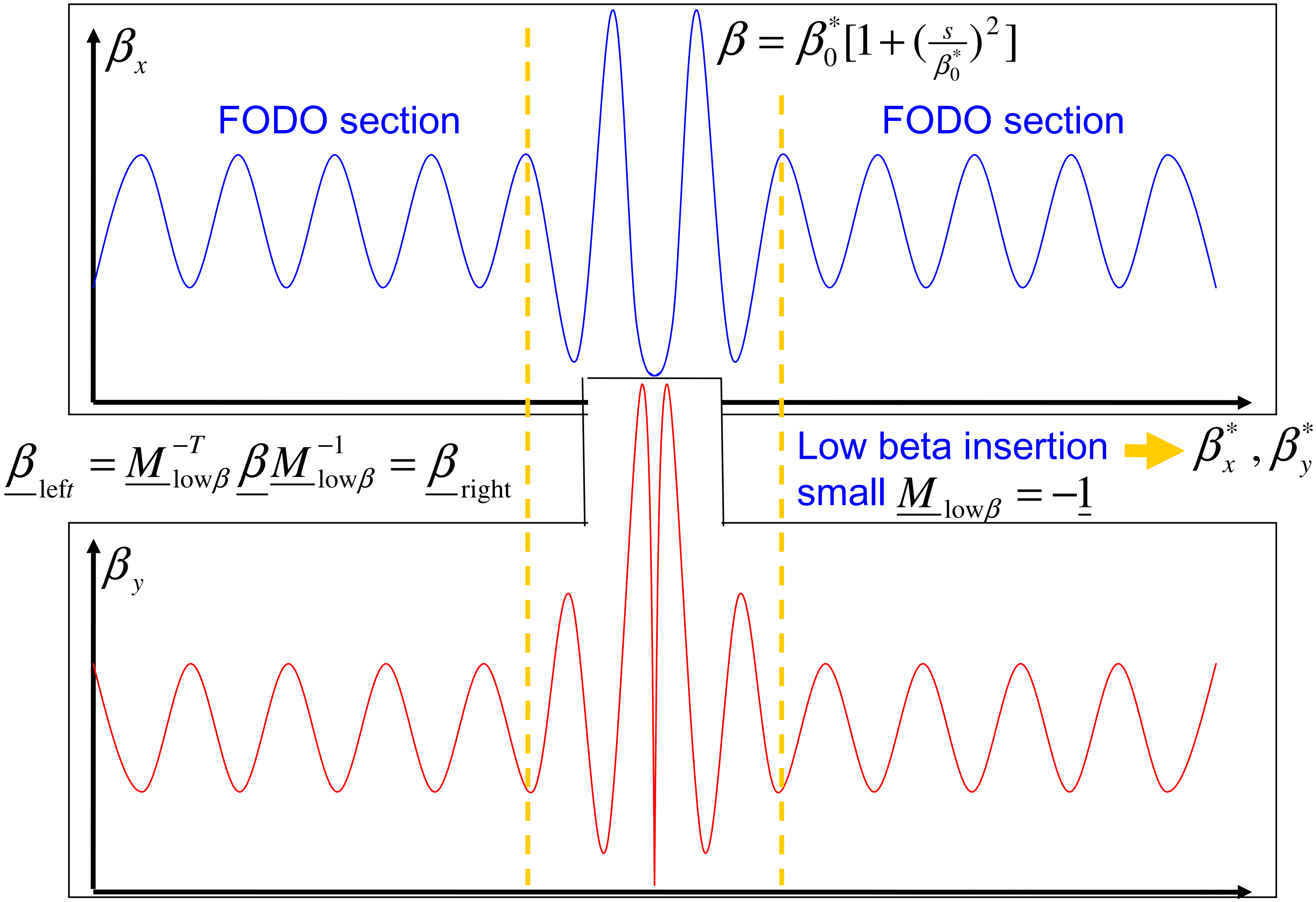

$$\beta = \beta_0^*[1+(\frac{s}{\beta_0^*})^2]$$

$$\underline{\beta}_{\text{left}} = \underline{M}_{\text{low}\beta}^{-T}\,\underline{\beta}\,\underline{M}_{\text{low}\beta}^{-1} = \underline{\beta}_{\text{right}}$$

Low beta insertion → β_x^*, β_y^*

small $\underline{M}_{\text{low}\beta} = -\underline{1}$

Quadrupole Errors

$$\vec{z}\,' = \underline{L}(s)\,\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)\,x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \left\{ \underline{M}(s) - \int_0^s \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix} \underline{M}(\hat{s},0)\,d\hat{s} \right\} \vec{z}_0$$

One quadrupole error:

$$\underline{M}(s,\hat{s}) + \Delta\underline{M}(s,\hat{s}) = \underline{M}(s,\hat{s}) - \underline{M}(s,\hat{s}) \begin{pmatrix} 0 & 0 \\ \Delta kl(\hat{s}) & 0 \end{pmatrix}$$

Quadrupole Error and Phase advance

CHESS & LEPP

$$\Delta \underline{M}(s,\hat{s}) = -\underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta k l(\hat{s}) & 0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\tilde{\psi} + \alpha_0 \sin\tilde{\psi}] & \sqrt{\beta_0\beta}\sin\tilde{\psi} \\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0 - \alpha)\cos\tilde{\psi} - (1+\alpha_0\alpha)\sin\tilde{\psi}] & \sqrt{\frac{\beta_0}{\beta}}[\cos\tilde{\psi} - \alpha\sin\tilde{\psi}] \end{pmatrix}$$

$$\Delta \underline{M}(s,\hat{s}) = -\Delta k l(\hat{s})\begin{pmatrix} \sqrt{\hat{\beta}\beta}\sin\psi & 0 \\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\psi - \alpha\sin\psi] & 0 \end{pmatrix} \quad , \qquad \tilde{\psi} = \psi - \hat{\psi}$$

$$= \begin{pmatrix} \frac{\frac{1}{2}\Delta\beta[\cos\psi + \hat{\alpha}\sin\psi] + \Delta\psi\beta[\hat{\alpha}\cos\psi - \sin\psi]}{\sqrt{\hat{\beta}\beta}} & \sqrt{\hat{\beta}}\left(\frac{\frac{\Delta\beta}{2}\sin\psi + \Delta\psi\beta\cos\psi}{\sqrt{\beta}}\right) \\ \cdots & \cdots \end{pmatrix}$$

$$\Delta\psi = -\frac{\Delta\beta}{2\beta}\tan\tilde{\psi}$$

$$\frac{1}{2}\Delta\beta\cos\tilde{\psi} + \frac{1}{2}\Delta\beta\frac{\sin^2\tilde{\psi}}{\cos\tilde{\psi}} = \frac{1}{2}\Delta\beta\frac{1}{\cos\tilde{\psi}} = -\Delta k l(\hat{s})\beta\hat{\beta}\sin\tilde{\psi}$$

Quadrupole Error correction

$$\Delta\psi = \Delta k l(\hat{s})\hat{\beta}\sin^2(\psi - \hat{\psi})$$

→ More focusing always increases the tune

$$\frac{\Delta\beta}{\beta} = -\Delta k l(\hat{s})\hat{\beta}\sin(2[\psi - \hat{\psi}])$$

→ Beta beat oscillates twice as fast as orbit.

$$\Delta\psi = \sum_j \Delta k l_j \beta_j \tfrac{1}{2}[1 - \cos(2[\psi - \psi_j])]$$

$$\frac{\Delta\beta}{\beta} = -\sum_j \Delta k l_j(\hat{s})\beta_j \sin(2[\psi - \psi_j])$$

When beta functions and betatron phases have been measured at many places, quadrupoles can be changed with these formulas to correct the Twiss errors.