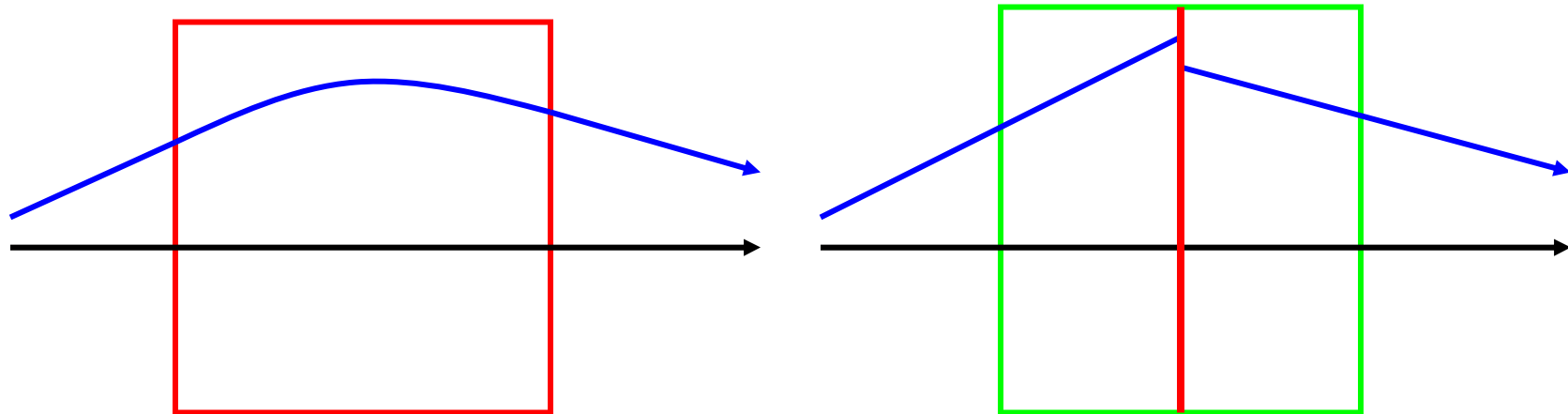


The Thin Lens Approximation



CHESS & LEPP



$$\vec{z}(s) = \underline{M}(s) \vec{z}_0 = \underline{D}\left(\frac{s}{2}\right) \underline{D}^{-1}\left(\frac{s}{2}\right) \underline{M}(s) \underline{D}^{-1}\left(\frac{s}{2}\right) \underline{D}\left(\frac{s}{2}\right) \vec{z}_0$$

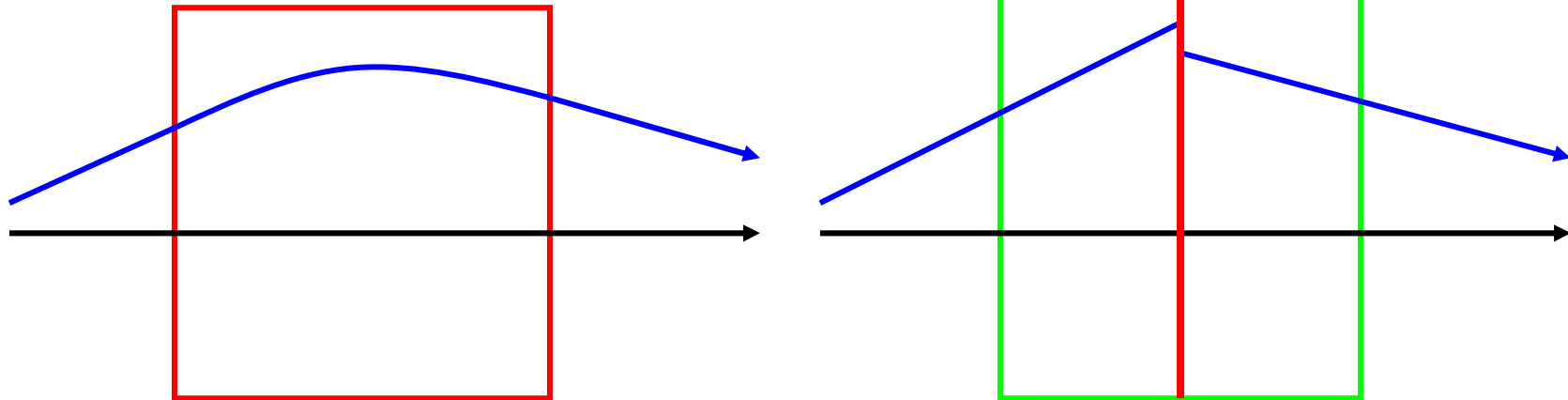
Drift: $\underline{M}_{\text{drift}}^{\text{thin}}(s) = \underline{D}^{-1}\left(\frac{s}{2}\right) \underline{D}(s) \underline{D}^{-1}\left(\frac{s}{2}\right) = \underline{1}$



The Thin Lens Quadrupole



CHESS & LEPP



$$\underline{M}_{\text{quad},x}^{\text{thin}}(s) = \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\sqrt{k}s) & \frac{1}{\sqrt{k}} \sin(\sqrt{k}s) \\ -\sqrt{k} \sin(\sqrt{k}s) & \cos(\sqrt{k}s) \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix}$$

$$\approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & s \\ -ks & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \approx \begin{pmatrix} 1 & -\frac{s}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{s}{2} \\ -ks & 1 + \frac{ks^2}{2} \end{pmatrix}$$

Weak magnet limit: $\sqrt{k}s \ll 1$

$$\underline{M}_{\text{quad},x}^{\text{thin}}(s) \approx \begin{pmatrix} 1 & 0 \\ -ks & 1 \end{pmatrix}$$



The Thin Lens Dipole



CHESS & LEPP

$$\underline{M} = \begin{pmatrix} \cos(\kappa s) & \frac{1}{\kappa} \sin(\kappa s) & \underline{0} & 0 & \kappa^{-1}[1 - \cos(\kappa s)] \\ -\kappa \sin(\kappa s) & \cos(\kappa s) & \underline{0} & 0 & \sin(\kappa s) \\ \underline{0} & \underline{0} & 1 & s & \underline{0} \\ -\sin(\kappa s) & \kappa^{-1}[\cos(\kappa s) - 1] & 0 & 1 & \underline{0} \\ 0 & 0 & \underline{0} & \underline{0} & 1 \end{pmatrix}$$

Weak magnet limit: $\kappa s \ll 1$

$$\underline{M}_{\text{bend},x\tau}^{\text{thin}}(s) = \underline{D}\left(-\frac{s}{2}\right) \underline{M}_{\text{bend},x\tau} \underline{D}\left(-\frac{s}{2}\right) \approx \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\kappa^2 s & 1 & 0 & \kappa s \\ -\kappa s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Thin Combined Function Bend



CHESS & LEPP

$$\underline{M}_6 = \begin{pmatrix} \underline{M}_x & \underline{0} & \vec{0} \vec{D} \\ \underline{0} & \underline{M}_y & \underline{0} \\ \underline{T} & \underline{0} & \underline{1} \end{pmatrix}$$

Weak magnet limit: $\kappa s \ll 1$

$$\underline{M}_x = \begin{pmatrix} \cos(\sqrt{K} s) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} s) \\ -\sqrt{K} \sin(\sqrt{K} s) & \cos(\sqrt{K} s) \end{pmatrix}$$

$$\underline{M}_x^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ -K s & 1 \end{pmatrix}$$

$$\underline{M}_y = \begin{pmatrix} \cosh(\sqrt{k} s) & \frac{1}{\sqrt{k}} \sinh(\sqrt{k} s) \\ \sqrt{k} \sinh(\sqrt{k} s) & \cosh(\sqrt{k} s) \end{pmatrix}$$

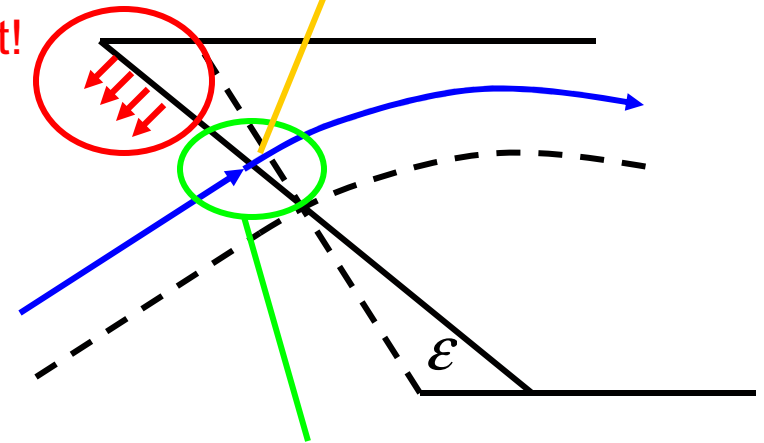
$$\underline{M}_y^{\text{thin}} = \begin{pmatrix} 1 & 0 \\ k s & 1 \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} \frac{\kappa}{K} [1 - \cos(\sqrt{K} s)] \\ \frac{\kappa}{\sqrt{K}} \sin(\sqrt{K} s) \end{pmatrix}$$

$$\vec{D} = \begin{pmatrix} 0 \\ \kappa s \end{pmatrix}$$



Fringe field has a horizontal field component!



Extra bending focuses!

Horizontal focusing with $\Delta x' = -x \frac{\tan(\epsilon)}{\rho}$

$$B_x = \partial_y B_s \Big|_{y=0} y \tan(\epsilon) = \partial_s B_y \Big|_{y=0} y \tan(\epsilon)$$

$$y'' = \frac{q}{p} \partial_s B_y \Big|_{y=0} y \tan(\epsilon)$$

$$\Delta y' = \int y'' ds = \frac{q}{p} B_y y \tan(\epsilon) = y \frac{\tan(\epsilon)}{\rho}$$

Quadrupole effect with

$$kl = \frac{\tan(\epsilon)}{\rho}$$

$$\vec{z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{\tan(\epsilon)}{\rho} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{\tan(\epsilon)}{\rho} & 1 \end{pmatrix} \vec{z}_0$$



Cyclotrons with edge focusing



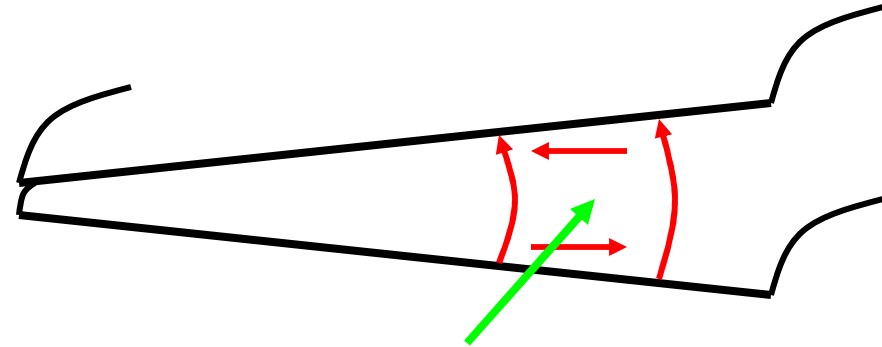
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- The isocyclotron with constant

$$\omega_z = \frac{q}{m_0 \gamma(E)} B_z(r(E))$$

Up to 600MeV but
this vertically defocuses the beam.

Edge focusing is therefore used.





Variation of Constants



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$$\vec{z}' = \vec{f}(\vec{z}, s)$$

$$\vec{z}' = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s) \quad \text{Field errors, nonlinear fields, etc can lead to } \Delta\vec{f}(\vec{z}, s)$$

$$\vec{z}'_H = \underline{L}(s)\vec{z}_H \Rightarrow \vec{z}_H(s) = \underline{M}(s)\vec{z}_{H0} \quad \text{with} \quad \underline{M}'(s)\vec{a} = \underline{L}(s)\underline{M}(s)\vec{a}$$

$$\vec{z}(s) = \underline{M}(s)\vec{a}(s) \Rightarrow \vec{z}'(s) = \underline{M}'(s)\vec{a} + \underline{M}(s)\vec{a}'(s) = \underline{L}(s)\vec{z} + \Delta\vec{f}(\vec{z}, s)$$

$$\vec{a}(s) = \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s}) d\hat{s}$$

$$\vec{z}(s) = \underline{M}(s) \left\{ \vec{z}_0 + \int_0^s \underline{M}^{-1}(\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s}) d\hat{s} \right\}$$

$$= \vec{z}_H(s) + \int_0^s \underline{M}(s - \hat{s})\Delta\vec{f}(\vec{z}(\hat{s}), \hat{s}) d\hat{s}$$

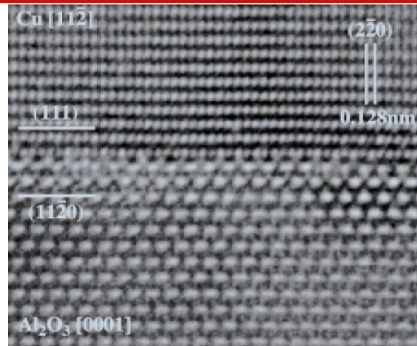
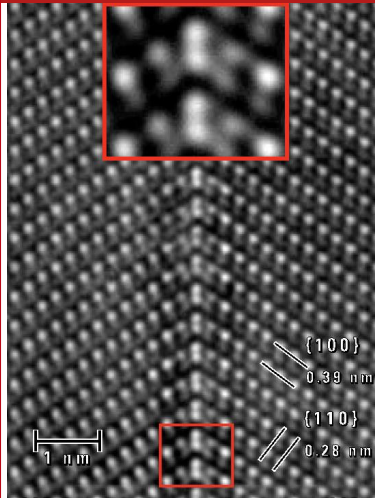
Perturbations are propagated
from s to s'



Aberration Correction



CHESS & LEPP



$$w_2(s) = w_H(s) + C(s)w_0^2\bar{w}_0 + \dots$$

$$w_2(s) = w_H(s) + A(s)\bar{w}_0^2 + B(s)w_0^2\bar{w}_0 + \dots$$

~~$$w_2(s) = w_H(s) + A(s)\bar{w}_0^2 + 2B(s)w_0^2\bar{w}_0 + \dots$$~~

2B cancels C!

Quadratic in
sextupole strength

Linear in
solenoid strength

