



Significance of Hamiltonian



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The equations of motion can be determined by one function:

$$\frac{d}{ds} x = \partial_{p_x} H(\vec{z}, s), \quad \frac{d}{ds} p_x = -\partial_x H(\vec{z}, s), \quad \dots$$

$$\frac{d}{ds} \vec{z} = \underline{J} \vec{\partial} H(\vec{z}, s) = \vec{F}(\vec{z}, s) \quad \text{with} \quad \underline{J} = \text{diag}(\underline{J}_2), \quad \underline{J}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The force has a **Hamiltonian Jacobi Matrix**:

A linear force:
$$\vec{F}(\vec{z}, s) = \underline{F}(s) \cdot \vec{z}$$

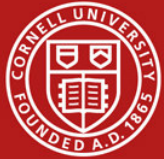
The **Jacobi Matrix** of a linear force: $\underline{F}(s)$

The general Jacobi Matrix :
$$F_{ij} = \partial_{z_j} F_i \quad \text{or} \quad \underline{F} = \left(\vec{\partial} \vec{F}^T \right)^T$$

Hamiltonian Matrices:
$$\underline{F} \underline{J} + \underline{J} \underline{F}^T = 0$$

Prove :
$$F_{ij} = \partial_{z_j} F_i = \partial_{z_j} J_{ik} \partial_{z_k} H = J_{ik} \partial_k \partial_j H \Rightarrow \underline{F} = \underline{J} \underline{D} \underline{H}$$

$$\underline{F} \underline{J} + \underline{J} \underline{F}^T = \underline{J} \underline{D} \underline{J} \underline{H} + \underline{J} \underline{D}^T \underline{J}^T \underline{H} = 0$$



H → Symplectic Flows



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The flow of a Hamiltonian equation of motion has a **symplectic Jacobi Matrix**

The **flow or transport map**: $\vec{z}(s) = \vec{M}(s, \vec{z}_0)$

A linear flow: $\vec{z}(s) = \underline{M}(s) \cdot \vec{z}_0$

The Jacobi Matrix of a linear flow: $\underline{M}(s)$

The general **Jacobi Matrix** : $M_{ij} = \partial_{z_{0j}} M_i$ or $\underline{M} = \left(\vec{\partial}_0 \vec{M}^T \right)^T$

The **Symplectic Group SP(2N)** : $\underline{M} \underline{J} \underline{M}^T = \underline{J}$

$$\frac{d}{ds} \vec{z} = \frac{d}{ds} \vec{M}(s, \vec{z}_0) = \underline{J} \vec{\nabla} H = \vec{F} \quad \frac{d}{ds} M_{ij} = \partial_{z_{0j}} F_i(\vec{z}, s) = \partial_{z_{0j}} M_k \partial_{z_k} F_i(\vec{z}, s)$$

$$\frac{d}{ds} \underline{M}(s, \vec{z}_0) = \underline{F}(\vec{z}, s) \underline{M}(s, \vec{z}_0)$$

$$\underline{K} = \underline{M} \underline{J} \underline{M}^T$$

$$\frac{d}{ds} \underline{K} = \frac{d}{ds} \underline{M} \underline{J} \underline{M}^T + \underline{M} \underline{J} \frac{d}{ds} \underline{M}^T = \underline{F} \underline{M} \underline{J} \underline{M}^T + \underline{M} \underline{J} \underline{M}^T \underline{F}^T = \underline{F} \underline{K} + \underline{K} \underline{F}^T$$

$\underline{K} = \underline{J}$ is a solution. Since this is a linear ODE, $\underline{K} = \underline{J}$ is the unique solution.