The comoving Coordinate System
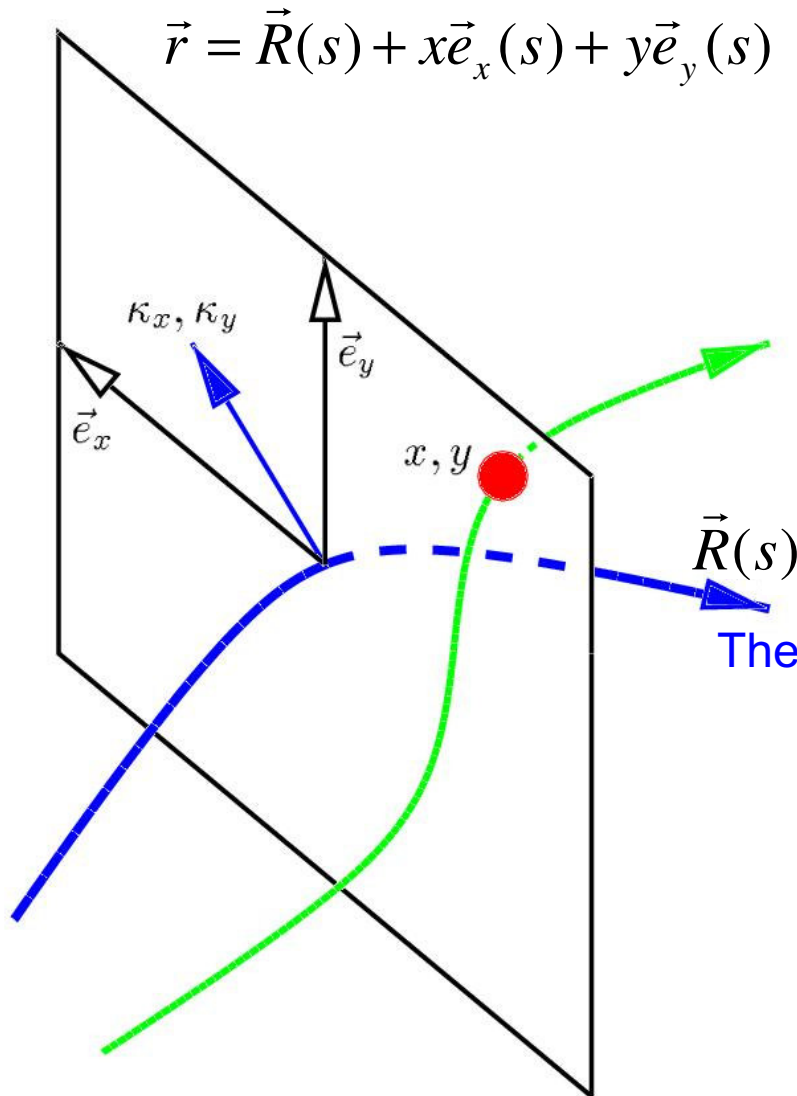
$$\vec{r} = \vec{R}(s) + x\vec{e}_x(s) + y\vec{e}_y(s)$$

$$\left|d\vec{R}\right| = ds$$

$$\vec{e}_s \equiv \frac{d}{ds}\vec{R}(s)$$

κ_x, κ_y

$\vec{e}_y$

$\vec{e}_x$

x, y

$\vec{R}(s)$

The time dependence of a particle's motion is often not as interesting as the trajectory along the accelerator length "s".

$$\frac{d^2}{dt^2}\vec{r} = \vec{f}_r(\vec{r}, \tfrac{d}{dt}\vec{r}, t)$$

3 dimensional ODE of 2nd order can be changed to a
6 dimensional ODE of 1st order:

$$\left. \begin{aligned} \frac{d}{dt}\vec{r} &= \frac{1}{m\gamma}\vec{p} = \frac{c}{\sqrt{p^2-(mc)^2}}\vec{p} \\ \frac{d}{dt}\vec{p} &= \vec{F}(\vec{r}, \vec{p}, t) \end{aligned} \right\} \quad \frac{d}{dt}\vec{Z} = \vec{f}_Z(\vec{Z}, t), \quad \vec{Z} = (\vec{r}, \vec{p})$$

If the force does not depend on time, as in a typical beam line magnet, the energy is conserved so that one can reduce the dimension to 5. The equation of motion is then autonomous.

Furthermore, the time dependence is often not as interesting as the trajectory along the accelerator length "s". Using "s" as the independent variable reduces the dimensions to 4. The equation of motion is then no longer autonomous.

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s), \quad \vec{z} = (x, y, p_x, p_y)$$
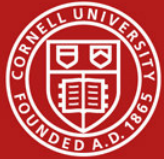
The 6D Equation of Motion

Usually one prefers to compute the trajectory as a function of "s" along the accelerator even when the energy is not conserved, as when accelerating cavities are in the accelerator.

Then the energy "E" and the time "t" at which a particle arrives at the cavities are important. And the equations become 6 dimensional again:

$$\frac{d}{ds}\vec{z} = \vec{f}_z(\vec{z}, s), \quad \vec{z} = (x, y, p_x, p_y, -t, E)$$

But: $\vec{z} = (\vec{r}, \vec{p})$ is an especially suitable variable, since it is a phase space vector so that its equation of motion comes from a Hamiltonian, or by variation principle from a Lagrangian.

$$\delta \int \left[p_x \dot{x} + p_y \dot{y} + p_s \dot{s} - H(\vec{r}, \vec{p}, t) \right] dt = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

$$\delta \int \left[p_x x' + p_y y' - H\, t' + p_s(x, y, p_x, p_y, t, H) \right] ds = 0 \quad \Rightarrow \quad \text{Hamiltonian motion}$$

The new canonical coordinates are: $\vec{z} = (x, y, p_x, p_y, -t, E)$ with $E = H$

The new Hamiltonian is: $\qquad\qquad K = -p_s(\vec{z}, s)$