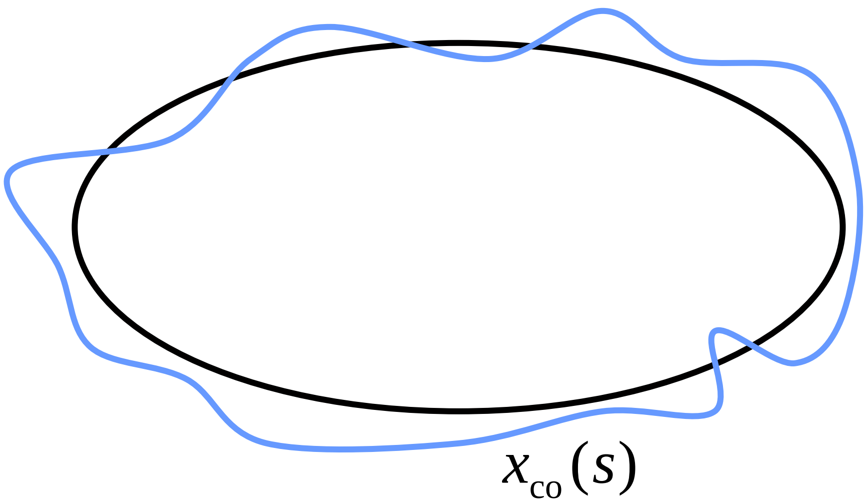
# Oscillations around a Closed Orbit

CHESS & LEPP




Particles oscillate around this periodic orbit, not around the design orbit.

$$\vec{z} = \vec{z}_\beta + \vec{z}_{co}$$

$$\vec{z}_\beta(L) + \vec{z}_{co}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \Delta\vec{z} = \underline{M}_0[\vec{z}_\beta(0) + \vec{z}_{co}(0)] + \Delta\vec{z}$$

$$= \underline{M}_0\vec{z}_\beta(0) + \vec{z}_{co}(L)$$

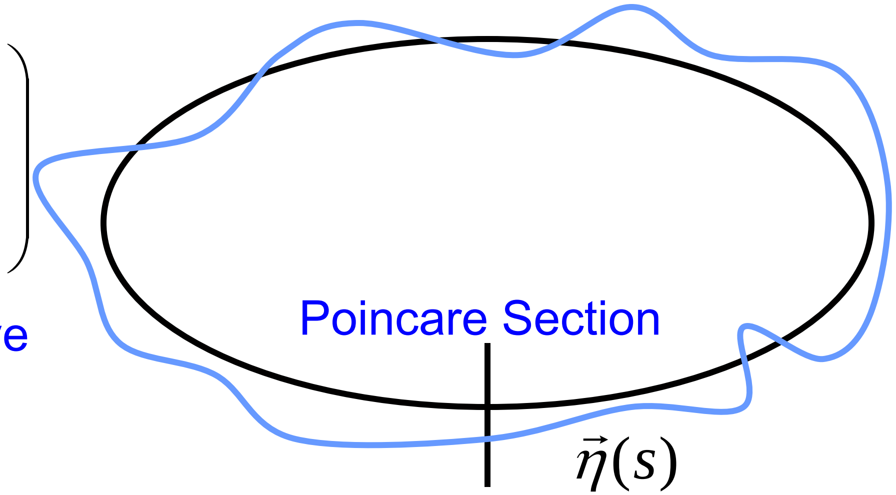
$$\boxed{\vec{z}_\beta(L) = \underline{M}_0\vec{z}_\beta(0)}$$

The closed orbit does not change the linear transport matrix.

# The Periodic Dispersion



$$\begin{pmatrix} \underline{M}_{0x}\vec{z}_0 + \vec{D}(L)\delta \\ M_{56}\delta \\ \delta \end{pmatrix} = \begin{pmatrix} \underline{M}_{0x} & \vec{0} & \vec{D}(L) \\ \vec{T}^T & 1 & M_{56} \\ \vec{0}^T & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{z}_0 \\ 0 \\ \delta \end{pmatrix}$$




The periodic orbit for particles with relative energy deviation δ is

$$\vec{\eta}(0) = \underline{M}_0\vec{\eta}(0) + \vec{D}(L) \qquad \vec{\eta}(L) = \underline{M}_{0x}\vec{\eta}(0) + \vec{D}(L) \quad \text{with} \quad \vec{\eta}(L) = \vec{\eta}(0)$$

$$\Downarrow$$

$$\boxed{\vec{\eta}(0) = [\underline{1} - \underline{M}_0(0)]^{-1}\vec{D}(L)}$$

Particles with energy deviation δ oscillates around this periodic orbit.

$$\vec{z} = \vec{z}_\beta + \delta\vec{\eta}$$

$$\vec{z}_\beta(L) + \delta\vec{\eta}(L) = \vec{z}(L) = \underline{M}_0\vec{z}(0) + \vec{D}(L)\delta = \underline{M}_0[\vec{z}_\beta(0) + \delta\vec{\eta}(0)] + \vec{D}(L)\delta$$

$$= \underline{M}_0\vec{z}_\beta(0) + \delta\vec{\eta}(L)$$

# Periodic dispersion Integral

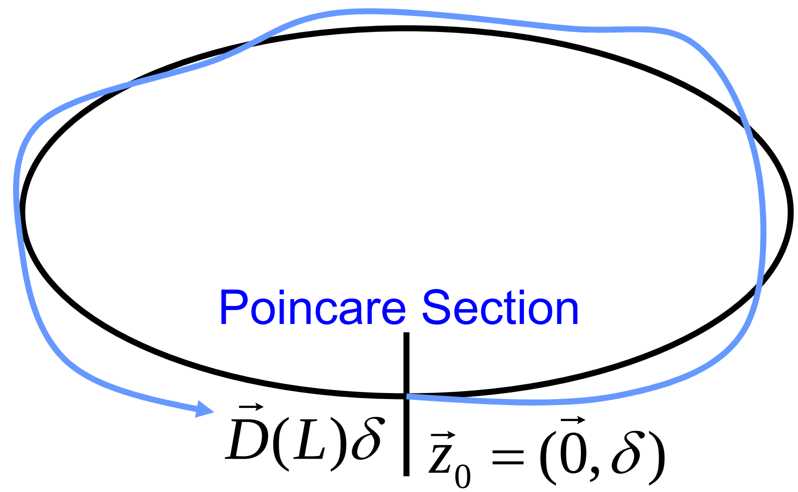


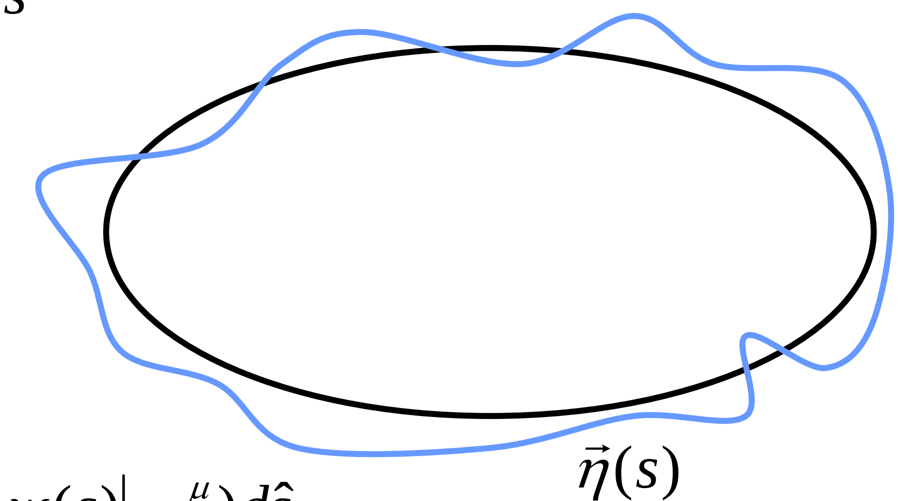
$$x' = a$$

$$a' = -(\kappa^2 + k)x + \kappa\delta$$

$$\vec{z} = \underline{M}\vec{z}_0 + \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$$

$$\Rightarrow \quad \vec{D}(L) = \int_0^L \underline{M}(L-\hat{s})\begin{pmatrix}0\\ \kappa(\hat{s})\end{pmatrix}ds'$$



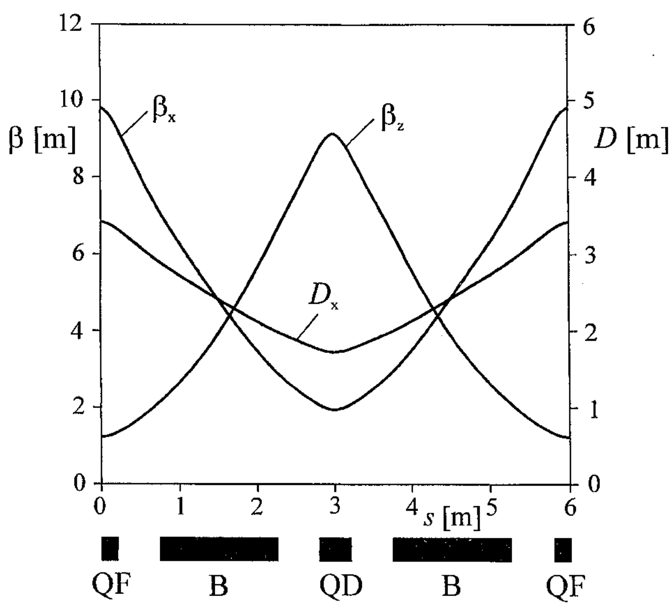

$$\Delta\kappa = \delta\kappa$$

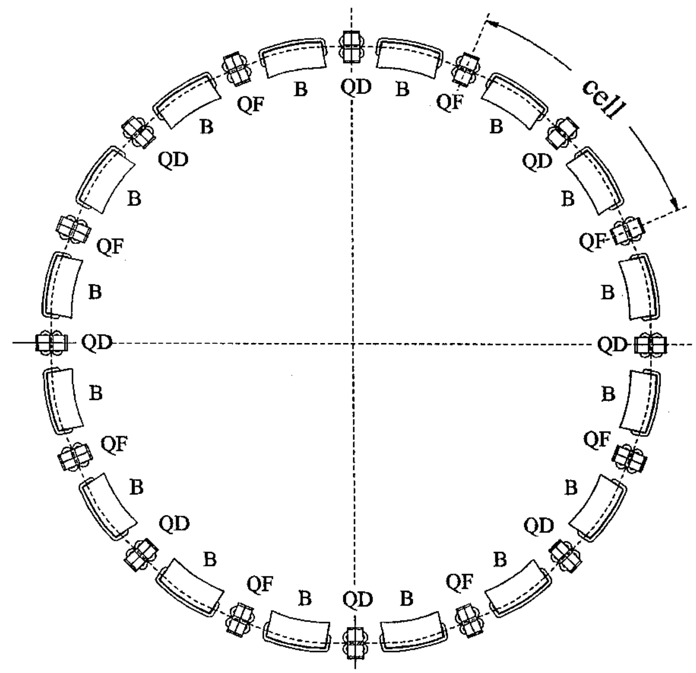
$$\eta(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}}\oint \kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(|\psi(\hat{s})-\psi(s)| - \tfrac{\mu}{2})d\hat{s}$$

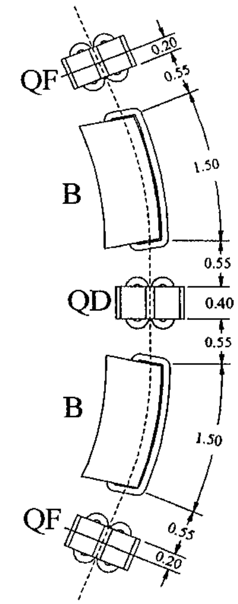
# The FODO Cell



Alternating gradients allow focusing in both transverse plains. Therefore focusing and defocusing quadrupoles are usually alternated and interleaved with bending magnets.




$$\underline{M}_0 = \underline{M}_{FoDo}^N$$

The periodic beta function and dispersion for each FODO is also periodic for the whole ring. Usually only large sections of the ring consist of FODOs.

# Thin Lens FODO Cell



$$\underline{M} \approx \underline{Q}^{\text{thin}}(\tfrac{kl}{2})\underline{D}(\tfrac{L}{2})\underline{Q}^{\text{thin}}(-kl)\underline{D}(\tfrac{L}{2})\underline{Q}^{\text{thin}}(\tfrac{kl}{2})$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1+\frac{kl}{2}\frac{L}{2} & \frac{L}{2} \\ -(\frac{kl}{2})^2\frac{L}{2} & 1-\frac{kl}{2}\frac{L}{2} \end{pmatrix}\begin{pmatrix} 1-\frac{kl}{2}\frac{L}{2} & \frac{L}{2} \\ -(\frac{kl}{2})^2\frac{L}{2} & 1+\frac{kl}{2}\frac{L}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1-2(\frac{kl}{2}\frac{L}{2})^2 & L(1+\frac{kl}{2}\frac{L}{2}) \\ -(\frac{kl}{2})^2 L(1-\frac{kl}{2}\frac{L}{2}) & 1-2(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix} \Rightarrow \begin{array}{l} \cos\mu_{FODO} = 1-2(\frac{kl}{2}\frac{L}{2})^2 \\ \beta = \left|\frac{L}{2\xi}\right|\sqrt{\frac{1+\xi}{1-\xi}} \\ \alpha = 0 \end{array}$$

$$\Rightarrow \xi = \frac{kl}{2}\frac{L}{2}\,,\quad \sin\frac{\mu_{FODO}}{2} = |\xi|$$

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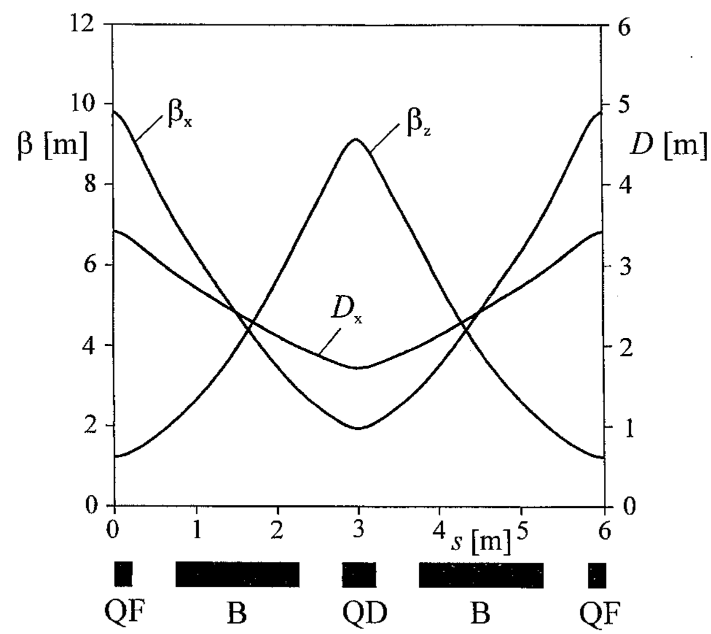
$$\alpha = 0$$




$$L_{FoDo} \approx 6\text{m}\,,\quad \varphi \approx 22.5°\,,\quad \mu_{FoDo} \approx \frac{\pi}{2}$$

$$\bar{\beta} \approx 3.8\text{m}$$

$$\beta_{\max} \approx 10.2\text{m}\,,\quad \beta_{\min} \approx 1.8\text{m}$$

# The Low Beta Insertion

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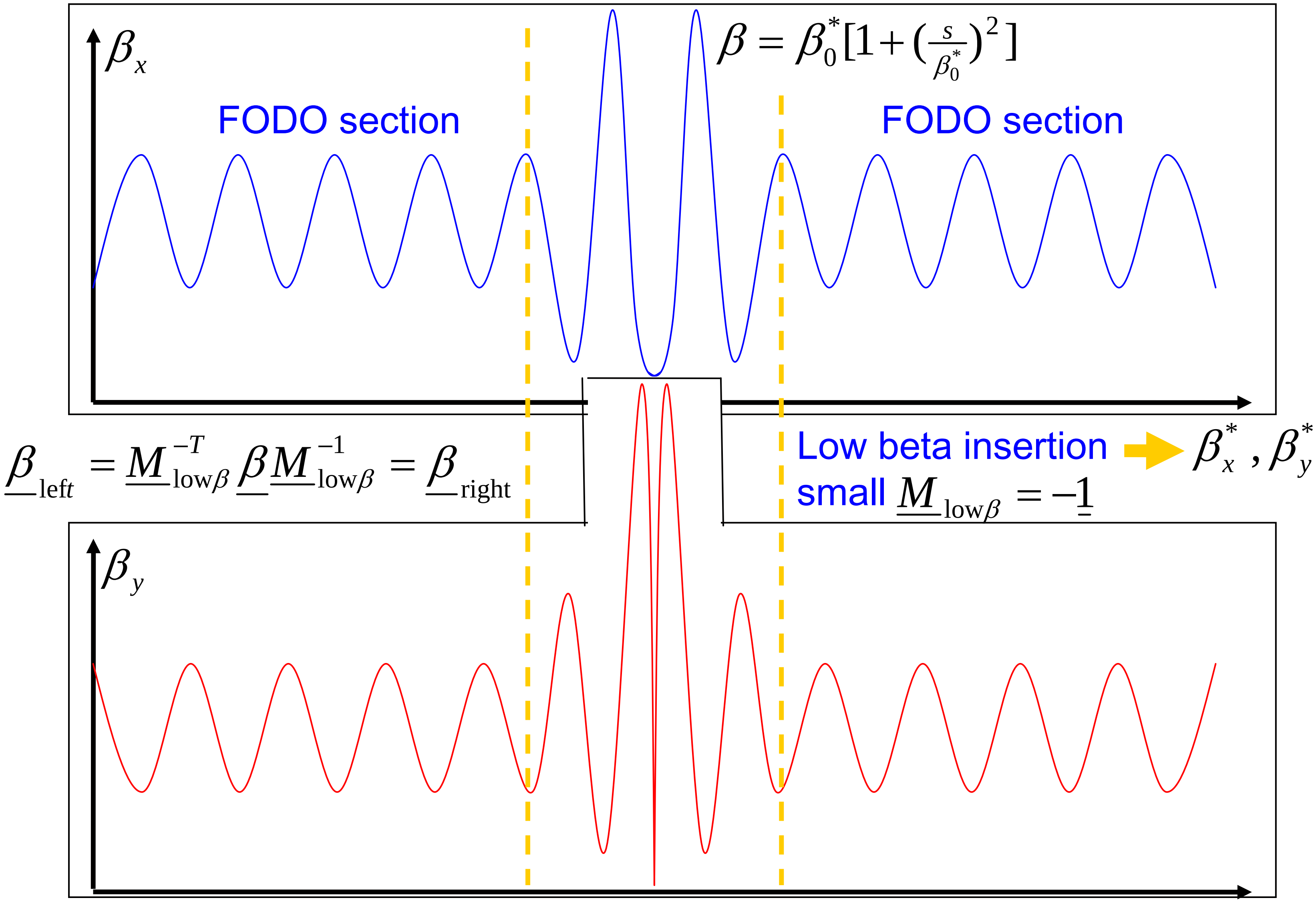


# Dispersion for a thin lens FODO Cell



$$\underline{M} \approx \underline{Q}^{\text{thin}}(\tfrac{kl}{2})\underline{D}(\tfrac{L}{4})[\vec{\varphi} + \underline{Q}^{\text{thin}}(-kl)\underline{D}(\tfrac{L}{4})\vec{\varphi}]$$

$$\vec{D} = \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix} + \underline{1}\right\}\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -\frac{kl}{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{L}{4} \\ 0 & 1 \end{pmatrix}\left\{\begin{pmatrix} 1+\frac{kl}{2}\frac{L}{2} & \frac{L}{2}(1+\frac{kl}{2}\frac{L}{4}) \\ kl & 1+\frac{kl}{2}\frac{L}{2} \end{pmatrix} + \underline{1}\right\}\begin{pmatrix} 0 \\ \varphi \end{pmatrix}$$

$$= \begin{pmatrix} 2+\frac{kl}{2}L & L(1+\frac{1}{2}\frac{kl}{2}\frac{L}{2}) \\ -(\frac{kl}{2})^2 L & 2-\frac{kl}{2}\frac{L}{2}-(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} L(1+\frac{1}{2}\frac{kl}{2}\frac{L}{2}) \\ 2-\frac{kl}{2}\frac{L}{2}-(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\varphi$$

$$\vec{\eta} = [\underline{1} - \underline{M}]^{-1}\vec{D} = \frac{1}{4(\frac{kl}{2}\frac{L}{2})^2}\begin{pmatrix} 2(\frac{kl}{2}\frac{L}{2})^2 & L(1+\frac{kl}{2}\frac{L}{2}) \\ -(\frac{kl}{2})^2 L(1-\frac{kl}{2}\frac{L}{2}) & 2(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\begin{pmatrix} L(1+\frac{1}{2}\frac{kl}{2}\frac{L}{2}) \\ 2-\frac{kl}{2}\frac{L}{2}-(\frac{kl}{2}\frac{L}{2})^2 \end{pmatrix}\varphi$$

$$= L\frac{1+\frac{1}{2}\xi}{2\xi^2}\begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \eta \\ \eta' \end{pmatrix}$$

# FODO Example



$$\xi = \frac{kl}{2}\frac{L}{2}$$

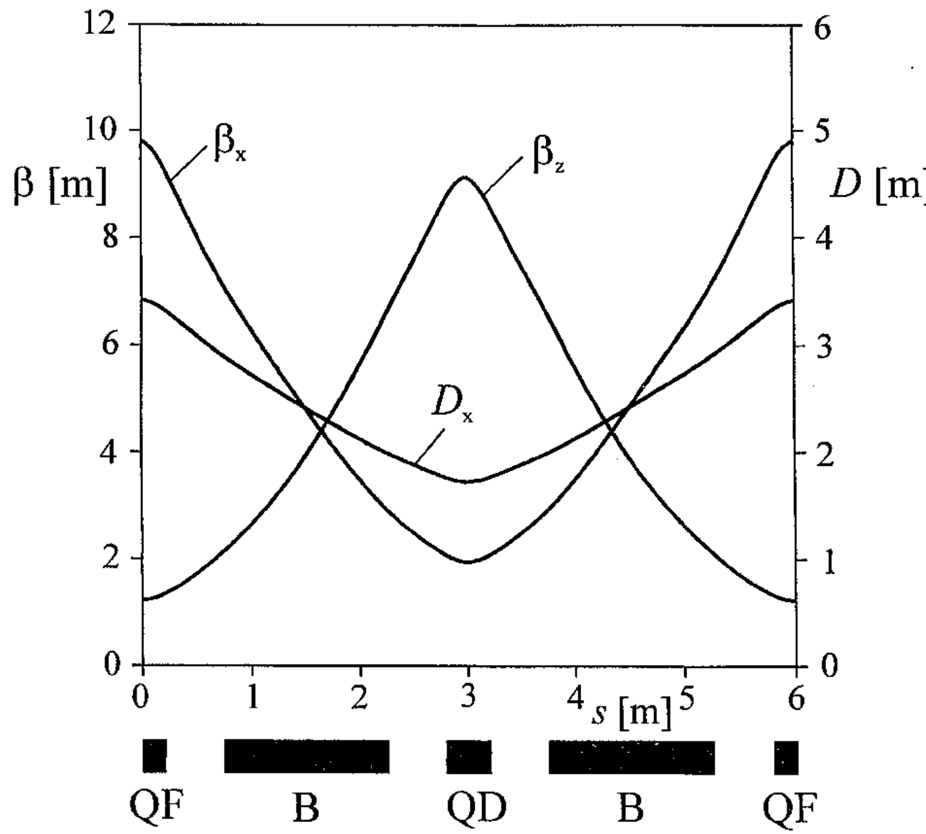
$$\sin\frac{\mu_{FODO}}{2} = |\xi|$$

$$\beta = \left|\frac{L}{2\xi}\right|\sqrt{\frac{1+\xi}{1-\xi}}$$

$$\alpha = 0$$

$$\eta = \frac{2+\xi}{(2\xi)^2}L\varphi$$

$$\eta' = 0$$




$$L_{FoDo} \approx 6\text{m}\,, \quad \varphi \approx 22.5^\circ\,, \quad \mu_{FoDo} \approx \tfrac{\pi}{2}$$

$$\bar\beta \approx 3.8\text{m}\,, \quad \bar\eta \approx 2\bar\beta\frac{\bar\beta}{\rho} \approx 3.8\text{m}$$

$$\beta_{\max} \approx 10.2\text{m}\,, \quad \beta_{\min} \approx 1.8\text{m}\,, \quad \eta_{\max} \approx 3.2\text{m}\,, \quad \eta_{\min} \approx 1.5\text{m}$$

# Quadrupole Errors



$$\vec{z}' = \underline{L}(s)\,\vec{z} + \Delta\vec{f}(\vec{z},s)$$

$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z},\hat{s})\,d\hat{s} \approx \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_H,\hat{s})\,d\hat{s}$$

$$x'' = -(\kappa^2 + k)x - \Delta k(s)\,x \quad \Rightarrow \quad \begin{pmatrix} x' \\ a' \end{pmatrix} = \begin{pmatrix} a \\ -(\kappa^2 + k)x \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \Delta k(s) & 0 \end{pmatrix}\begin{pmatrix} x \\ a \end{pmatrix}$$

$$\vec{z}(s) = \underline{M}(s)\vec{z}_0 - \int_0^s \underline{M}(s,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix}\underline{M}(\hat{s})\vec{z}_0\,d\hat{s}$$

$$\underline{M}_0(s) + \Delta\underline{M}_0(s) = \underline{M}_0(s) - \int_s^{s+L} \underline{M}(s+L,\hat{s})\begin{pmatrix} 0 & 0 \\ \Delta k(\hat{s}) & 0 \end{pmatrix}\underline{M}(\hat{s},s)\,d\hat{s}$$

# Quadrupole Error and Tune Shift



$$\underline{M}_0(s)+\Delta\underline{M}_0(s)=\underline{M}_0(s)-\int_s^{s+L}\underline{M}(s+L,\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\underline{M}(\hat{s},s)d\hat{s}$$

$$\cos(\mu+\Delta\mu)=\cos\mu-\tfrac{1}{2}\int_s^{s+L}Tr\left[\underline{M}(s+L,\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\underline{M}(\hat{s},s)\right]d\hat{s}$$

$$=\cos\mu-\tfrac{1}{2}\int_s^{s+L}Tr\left[\underline{M}_0(\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\right]d\hat{s}$$

$$=\cos\mu-\tfrac{1}{2}\int_0^{L}\Delta k(\hat{s})\beta(\hat{s})\,d\hat{s}\sin\mu\approx\cos\mu-\Delta\mu\sin\mu$$

$$\Delta\mu=\tfrac{1}{2}\int_0^{L}\Delta k(\hat{s})\beta(\hat{s})\,ds$$

One quadrupole error:

$$\Delta\nu=\tfrac{\beta}{4\pi}\Delta k$$

→ More focusing always increases the tune

# Quadrupole Error and Beta Beat



$$\underline{M}_0(s)+\Delta\underline{M}_0(s)=\underline{M}_0(s)-\int_s^{s+L}\underline{M}(s+L,\hat{s})\begin{pmatrix}0 & 0\\ \Delta k(\hat{s}) & 0\end{pmatrix}\underline{M}(\hat{s},s)d\hat{s}$$

$$(\beta+\Delta\beta)\sin(\mu+\Delta\mu)=\beta\sin\mu-\int_s^{s+L}\Delta\hat{k}\beta\hat{\beta}\sin(\mu+\psi-\hat{\psi})\sin(\hat{\psi}-\psi)d\hat{s}$$

$$\approx\beta\sin\mu+\Delta\beta\sin\mu+\Delta\mu\beta\cos\mu$$

$$\Delta\beta=-\frac{1}{2\sin\mu}\int_s^{s+L}\Delta\hat{k}\beta\hat{\beta}[2\sin(\mu+\psi-\hat{\psi})\sin(\hat{\psi}-\psi)+\cos\mu]d\hat{s}$$

$$=\frac{\beta}{2\sin\mu}\int_s^{s+L}\Delta\hat{k}\hat{\beta}\cos(2[\hat{\psi}-\psi]-\mu)d\hat{s}=-\frac{\beta}{2\sin\mu}\int_0^{L}\Delta\hat{k}\hat{\beta}\cos(2|\hat{\psi}-\psi|-\mu)d\hat{s}$$

One quadrupole error:

$$\Delta\beta=-\frac{\beta}{2\sin\mu}\Delta\hat{k}\hat{\beta}\cos(2|\hat{\psi}-\psi|-\mu)$$

Focusing can increase or decrease the beta function

$$\frac{\Delta\beta_{\max}}{\beta}=2\pi\frac{\Delta\nu}{\sin\mu}$$

# Weakly nonlinear systems



Weakly nonlinear ODEs $\frac{d}{ds}\vec{z} = \vec{f}(\vec{z}, s)$

Have a right hand side that can be

approximated well by a truncated Taylor expansion

$$\vec{f}(\vec{z}, s) \approx \underline{L}(s)\,\vec{z} + \sum_{j,k} \vec{f}_{jk} z_j z_k + \sum_{j,k,l} \vec{f}_{jkl} z_j z_k z_l + \ldots + \sum_{\vec{k},\text{order O}} \vec{f}_{\vec{k}} \vec{z}^{\vec{k}} + \ldots$$

$$\vec{z}^{\vec{k}} = \prod_{n=1}^{N} z_n^{k_n}, \quad \sum_{\vec{k},\text{order O}} \ldots = \sum_{n=1}^{N} \sum_{k_n} \ldots \quad \text{with} \quad \sum_{n=1}^{N} k_n = O$$

By solving the Taylor expanded ODE one tries to find a Taylor expansion of the transport map: $\vec{M}(s; s_0, \vec{z}_0) \approx \underline{M}(s, s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order O}} \vec{M}_{\vec{k}} \vec{z}_0^{\vec{k}} + \ldots$

Note:

While this approach is usually chosen, it is not certain that a transport map of the Taylor expanded ODE is a Taylor expansion of the transport map of the original ODE. One therefore often speaks of “formally” finding the Taylor expansion of the transport map.

# Aberrations



$$\vec{M}(s;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order } O} \vec{M}_{\vec{k}} \vec{z}_0^{\,\vec{k}} + \ldots$$

The Taylor coefficients are called aberrations of order O and are denoted by

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}) \equiv M_{\vec{k},i} \,, \quad \text{order} \quad O = \sum_{n=1}^{6} k_n$$

$$\tfrac{d}{ds}\vec{z} = \vec{f}(\vec{z},s) \quad \Rightarrow \quad \vec{z}(s) = \vec{M}(s;s_0,\vec{z}_0)$$

$$\vec{M}(s;s_0,\vec{z}_0) \approx \underline{M}(s,s_0)\vec{z}_0 + \ldots + \sum_{\vec{k},\text{order } O} \vec{M}_{\vec{k}}^{\,`n} \vec{z}_0^{\,\vec{k}}$$

$$(z_i, z_1^{k_1} \ldots z_6^{k_6}) \equiv M_{\vec{k},i} \,, \quad \text{order} \quad O = \sum_{j=1}^{6} k_j$$

How can all these Taylor coefficients be computed?

# Iteration for Aberrations



$$\vec{z}(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_1(s) = \vec{z}_H(s)$$

$$\vec{z}_2(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vdots$$

$$\vec{z}_n(s) = \vec{z}_H(s) + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}(\vec{z}_{n-1}(\hat{s}),\hat{s})\,d\hat{s}$$

Taylor expansions: $\Delta\vec{f}(\vec{z},s) = \Delta\vec{f}_2(\vec{z},s) + \Delta\vec{f}_3(\vec{z},s) + \dots\ ,\quad \Delta f_O = \sum_{\vec{k},\text{order } O} \vec{f}_{\vec{k}}\vec{z}^{\vec{k}}$

$$\vec{z}_1(s) = \underline{M}(s)\vec{z}_0$$

$$\vec{z}_2(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\Delta\vec{f}_2(\vec{z}_1(\hat{s}),\hat{s})\,d\hat{s}$$

$$\vec{z}_3(s) = \underline{M}(s)\vec{z}_0 + \int_0^s \underline{M}(s,\hat{s})\{[\Delta\vec{f}_2(\vec{z}_2(\hat{s}),\hat{s})]_3 + \Delta\vec{f}_3(\vec{z}_1(\hat{s}),\hat{s})\}\,d\hat{s}$$

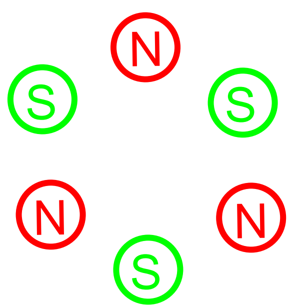
$$\vdots$$

# Sextupoles (revisited)



$$\psi = \Psi_3 \operatorname{Im}\{(x-iy)^3\} = \Psi_3 \cdot (y^3 - 3x^2 y) \quad \Rightarrow \quad \vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$C_3$ Symmetry




i) Sextupole fields hardly influence the particles close to the center, where one can linearize in x and y.

ii) In linear approximation a by Δx shifted sextupole has a quadrupole field.

iii) When Δx depends on the energy, one can build an energy dependent quadrupole.

$$\vec{B} = -\vec{\nabla}\psi = \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

$$x \mapsto \Delta x + x$$

$$\vec{B} \approx \Psi_3\, 3 \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix} + 6\Psi_3 \Delta x \begin{pmatrix} y \\ x \end{pmatrix} + O(\Delta x^2)$$

$$k_2 = 3!\,\Psi_3 \Rightarrow k_1 = k_2 \Delta x$$

# Chromaticity and its Correction



Chromaticity $\xi$ = energy dependence of the tune

$$\upsilon(\delta) = \upsilon + \tfrac{\partial \upsilon}{\partial \delta}\delta + \dots$$

$$\xi = \tfrac{\partial \upsilon}{\partial \delta} \quad \text{with} \quad \upsilon = \tfrac{\mu}{2\pi}$$

Natural chromaticity $\xi_0$ = energy dependence of the tune due to quadrupoles only

$$\xi_{x0} = -\frac{1}{4\pi}\oint \beta_x(\hat{s})k_1(\hat{s})d\hat{s} \qquad \xi_{y0} = \frac{1}{4\pi}\oint \beta_y(\hat{s})k_1(\hat{s})d\hat{s}$$

Particles with energy difference oscillate around the periodic dispersion leading to a quadrupole effect in sextupoles that also shifts the tune:

$$\xi_x = \frac{1}{4\pi}\oint \beta_x(-k_1 + \eta_x k_2)d\hat{s} \qquad \xi_y = \frac{1}{4\pi}\oint \beta_y(k_1 - \eta_y k_2)d\hat{s}$$

Typically the the chormaticity $\xi$ is chosen to be slightly positive, between 0 and 3.