Beta Function and Betatron Phase

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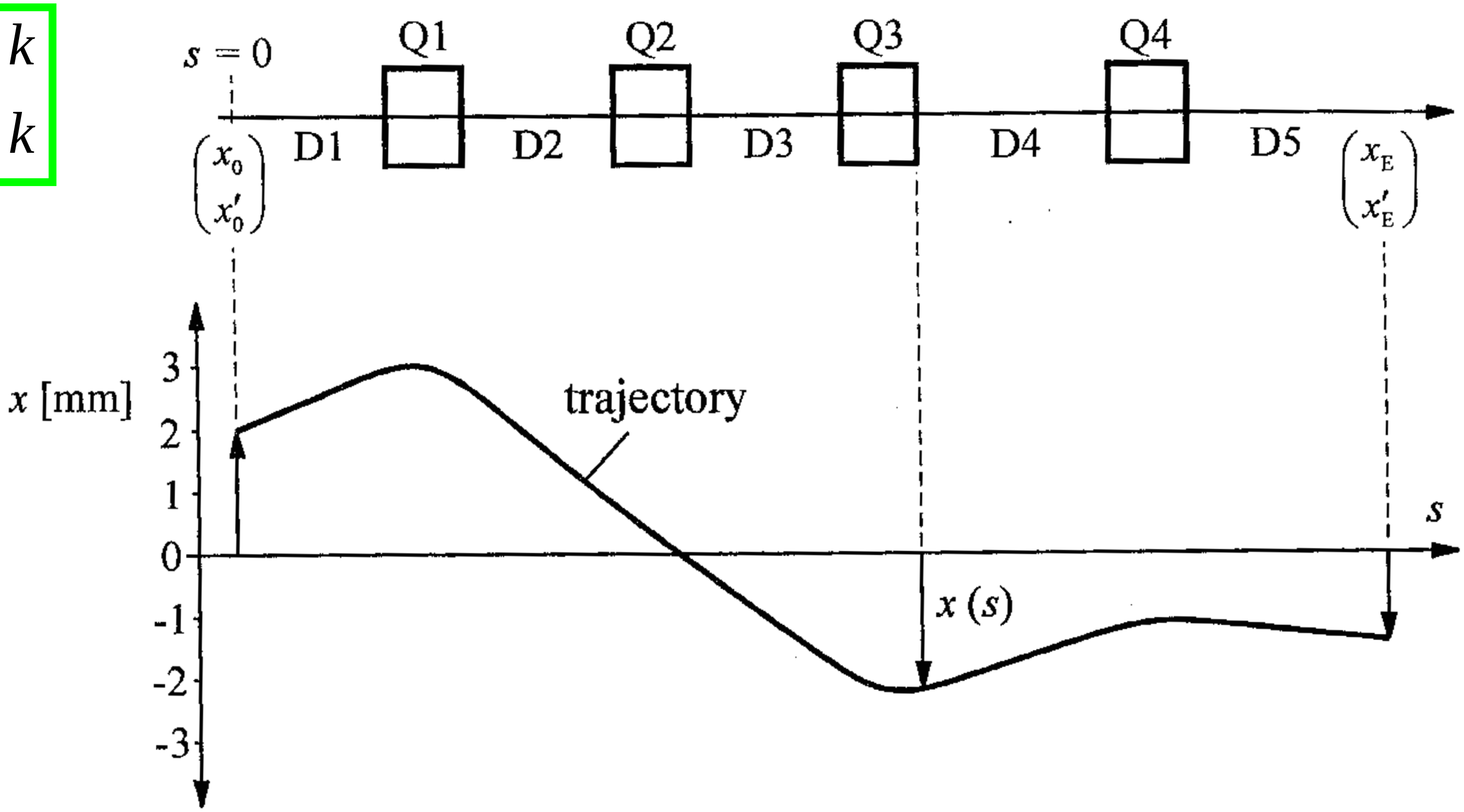
$$x'' = -x\,k$$
$$y'' = \;\; y\,k$$



$$x(s) = M_{11}(s)x_0 + M_{12}(s)x_0'$$
$$x(s) = \sqrt{2J\beta(s)}\sin(\psi(s)+\phi_0)$$

Twiss Parameters

$$x'' = -k\, x$$

$$x(s) = \sqrt{2J\beta(s)}\, \sin(\psi(s) + \phi_0)$$

$$x'(s) = \sqrt{\tfrac{2J}{\beta}}[\beta\psi' \cos(\psi(s) + \phi_0) - \alpha \sin(\psi(s) + \phi_0)] \quad \text{with} \quad \alpha = -\tfrac{1}{2}\beta'$$

$$x''(s) = \sqrt{\tfrac{2J}{\beta}}[(\beta\psi'' - 2\alpha\psi')\cos(\psi(s) + \phi_0) - (\alpha' + \tfrac{\alpha^2}{\beta} + \beta\psi'^2)\sin(\psi(s) + \phi_0)]$$

$$= \sqrt{\tfrac{2J}{\beta}}[-k\beta \sin(\psi(s) + \phi_0)]$$

$$\beta\psi'' - 2\alpha\psi' = \beta\psi'' + \beta'\psi' = (\beta\psi')' = 0 \quad \Rightarrow \quad \psi' = \tfrac{1}{\beta}$$

$$\alpha' + \gamma = k\beta \quad \text{with} \quad \gamma = \tfrac{1+\alpha^2}{\beta}$$

$$\beta' = -2\alpha$$

$$\alpha' = k\beta - \gamma$$

$$\psi = \int_0^s \frac{1}{\beta(s')} ds'$$

$\alpha, \beta, \gamma, \psi$ are called Twiss parameters.

What are the initial conditions?

Phase Space Ellipse

Particles with a common J and different ϕ all lie on an ellipse in phase space:

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J}\begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}\begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

(Linear transform of a circle)

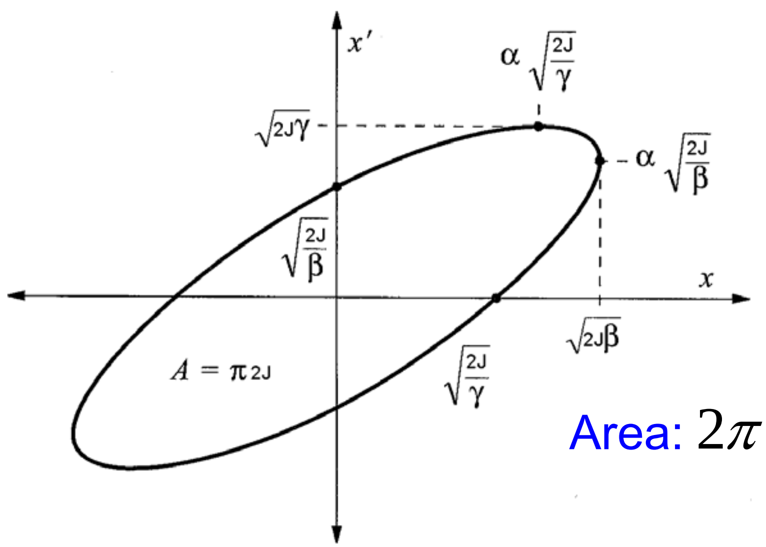
$$x_{\max} = \sqrt{2J\beta} \text{ at } x' = -\alpha\sqrt{\tfrac{2J}{\beta}}$$

$$(x,x')\begin{pmatrix} \frac{1}{\sqrt{\beta}} & \frac{\alpha}{\sqrt{\beta}} \\ 0 & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = (x,x')\begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}\begin{pmatrix} x \\ x' \end{pmatrix} = 2J$$

(Quadratic form)

$$\beta\gamma - \alpha^2 = 1$$

Area: $2\pi J$

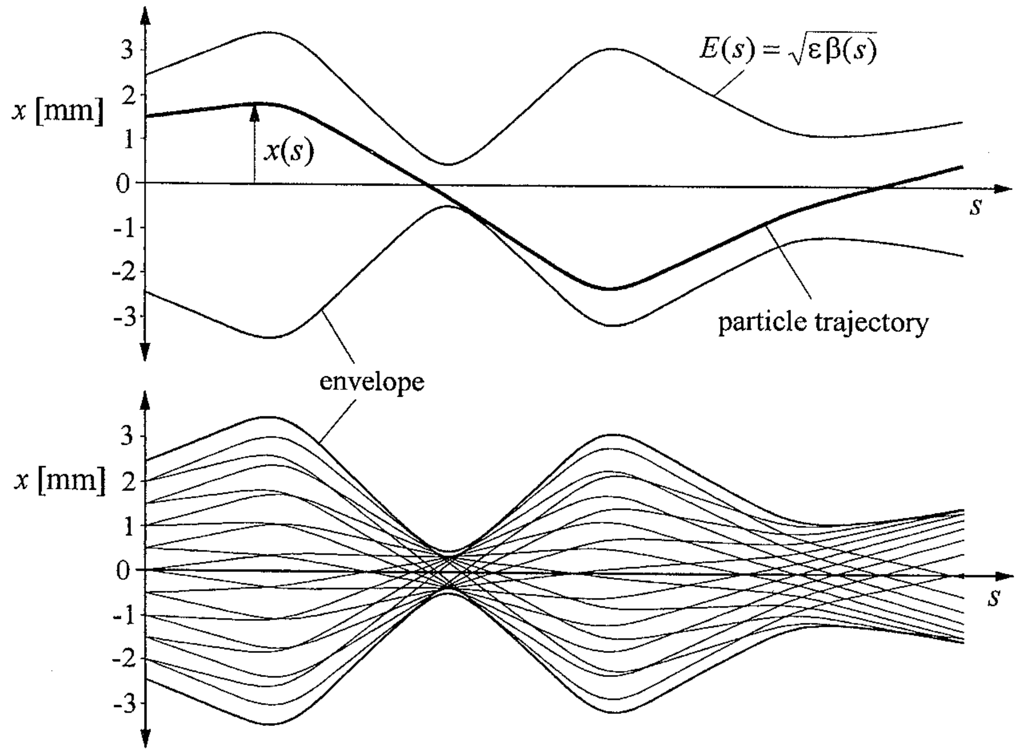


What β is for x, γ is for x'

$$x'_{\max} = \sqrt{2J\gamma} \text{ at } x = -\alpha\sqrt{\tfrac{2J}{\gamma}}$$

Area: $2\pi J$ → $\int_0^{2\pi}\int_0^{J} dJ d\phi = 2\pi J = \iint dx dx'$

The Beam Envelope



$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

In any beam there is a distribution of initial parameters. If the particles with the largest J are distributed in ϕ over all angles, then the envelope of the beam is described by $\sqrt{2J_{\max}\beta(s)}$

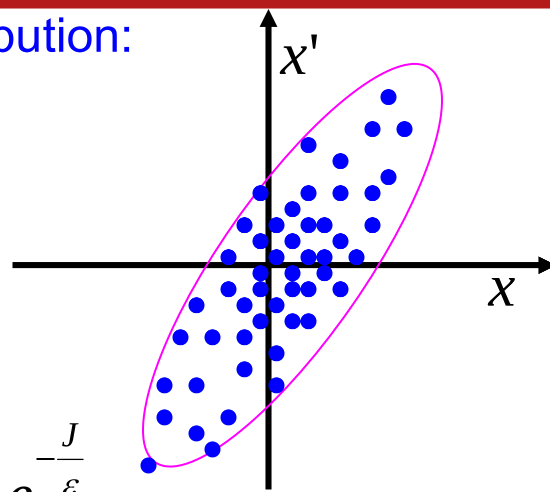
The initial conditions of β and α are chosen so that this is approximately the case.

Phase Space Distribution

Often one can fit a Gauss distribution to the particle distribution:

$$\rho(x,x') = \frac{1}{2\pi\varepsilon} e^{-\frac{\gamma x^2 + 2\alpha x x' + \beta x'^2}{2\varepsilon}}$$

The equi-density lines are then ellipses. And one chooses the starting conditions for β and α according to these ellipses!



$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix} \qquad \rho(J,\phi_0) = \frac{1}{2\pi\varepsilon} e^{-\frac{J}{\varepsilon}}$$

$$\langle 1 \rangle = \frac{1}{2\pi\varepsilon} \int_0^{2\pi} \int_0^{\infty} e^{-J/\varepsilon} dJ d\phi_0 = 1$$

Initial beam distribution ⟶ initial α, β, γ

$$\langle x^2 \rangle = \frac{1}{2\pi\varepsilon} \iint 2J\beta \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\beta \quad \longrightarrow \quad \langle x'^2 \rangle = \varepsilon\gamma$$

$$\langle x x' \rangle = -\frac{1}{2\pi\varepsilon} \iint 2J\alpha \sin\phi_0^2 e^{-J/\varepsilon} dJ d\phi_0 = \varepsilon\alpha$$

$$\varepsilon = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2}$$ is called the emittance.

Invariant of Motion

$$x(s) = \sqrt{2J\beta(s)}\,\sin(\psi(s) + \phi_0)$$

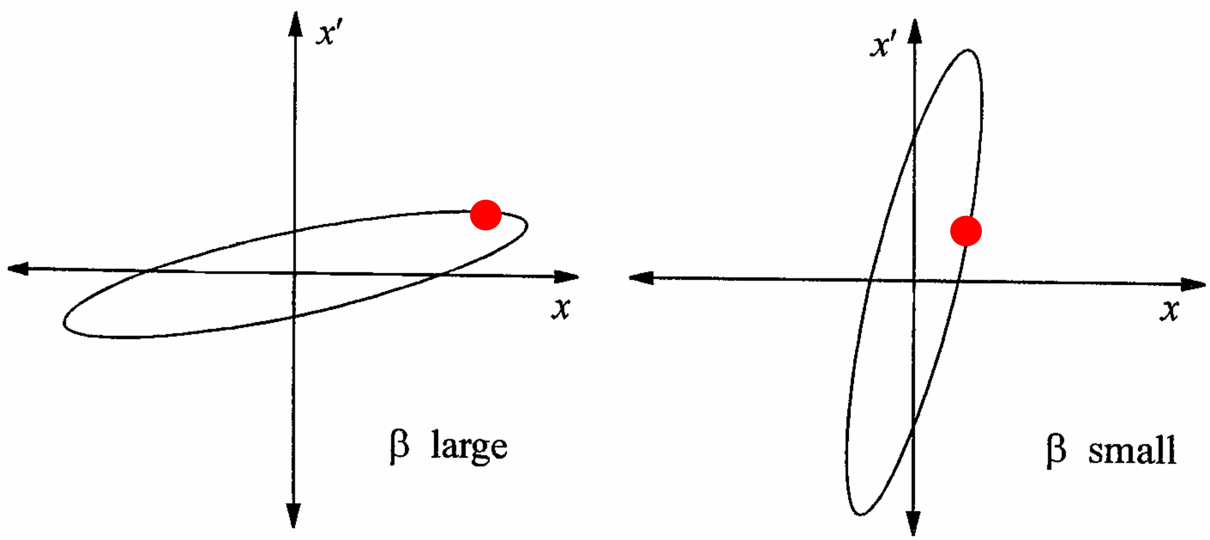
Where J and ϕ are given by the starting conditions x_0 and x'_0.

$$\gamma x^2 + 2\alpha x x' + \beta x'^2 = 2J$$

Leads to the invariant of motion:

$$f(x, x', s) = \gamma(s)x^2 + 2\alpha(s)xx' + \beta(s)x'^2 \quad \Rightarrow \quad \tfrac{d}{ds} f = 0$$

It is called the Courant-Snyder invariant.



Propagation of Twiss Parameters

$$(x_0, x_0')\begin{pmatrix}\gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0\end{pmatrix}\begin{pmatrix}x_0 \\ x_0'\end{pmatrix} = 2J$$

$$(x, x')\begin{pmatrix}\gamma & \alpha \\ \alpha & \beta\end{pmatrix}\begin{pmatrix}x \\ x'\end{pmatrix} = 2J = (x_0, x_0')\underline{M}^T\begin{pmatrix}\gamma & \alpha \\ \alpha & \beta\end{pmatrix}\underline{M}\begin{pmatrix}x_0 \\ x_0'\end{pmatrix}$$

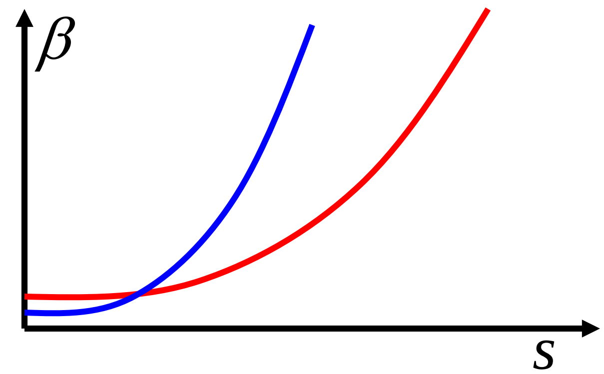
$$\begin{pmatrix}\gamma & \alpha \\ \alpha & \beta\end{pmatrix} = \underline{M}^{-T}\begin{pmatrix}\gamma_0 & \alpha_0 \\ \alpha_0 & \beta_0\end{pmatrix}\underline{M}^{-1}$$

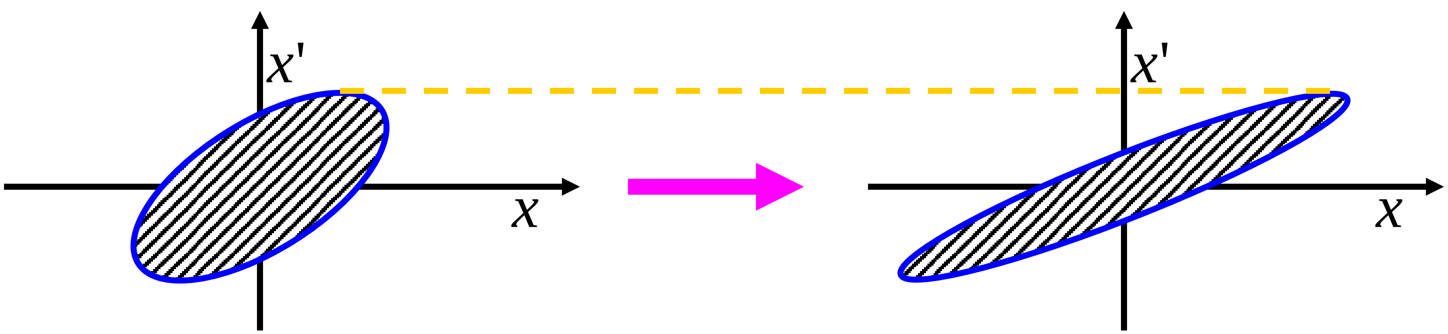
$$\boxed{\begin{pmatrix}\beta & -\alpha \\ -\alpha & \gamma\end{pmatrix} = \underline{M}\begin{pmatrix}\beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0\end{pmatrix}\underline{M}^T}$$

Twiss Parameters in a Drift

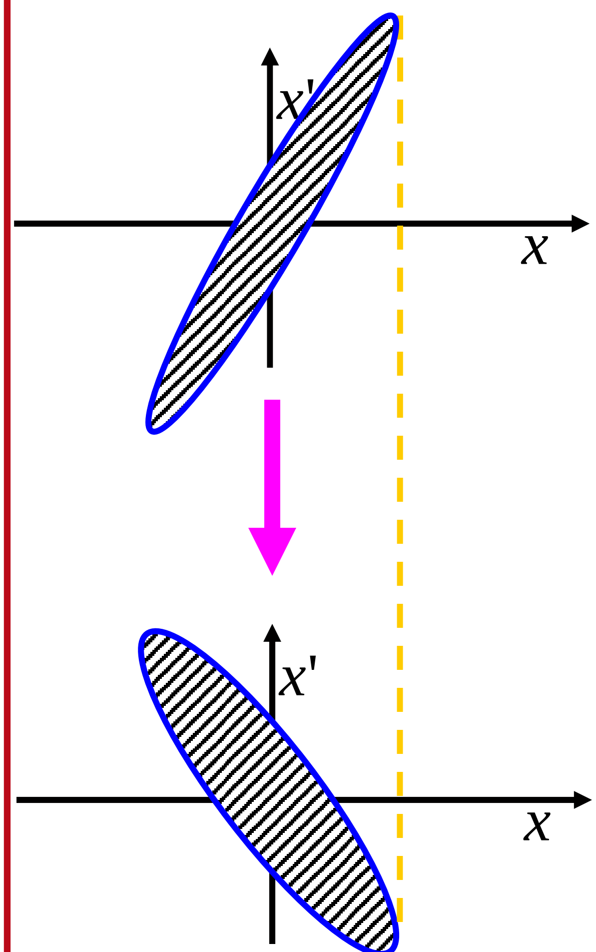
$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} = \begin{pmatrix} \beta_0 - 2\alpha_0 s + \gamma_0 s^2 & \gamma_0 s - \alpha_0 \\ \gamma_0 s - \alpha_0 & \gamma_0 \end{pmatrix}$$

$$\beta = \beta_0^*[1 + (\tfrac{s}{\beta_0^*})^2] \quad \text{for} \quad \alpha_0^* = 0$$



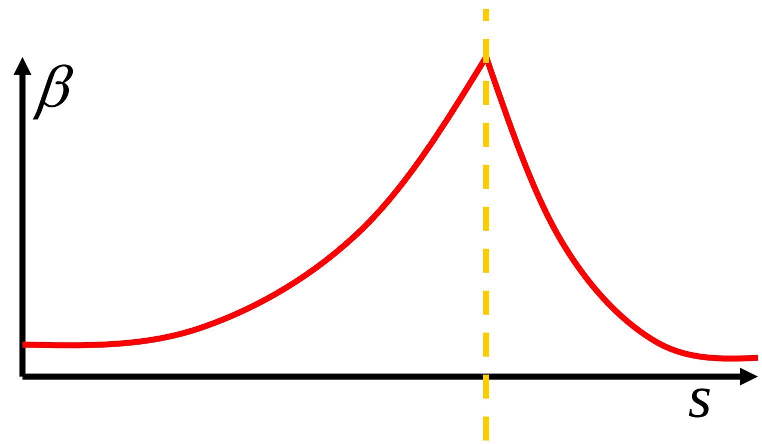


Twiss Parameters after a thin a Quadrupole



$$\begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} \beta_0 & -\alpha_0 \\ -\alpha_0 & \gamma_0 \end{pmatrix} \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\alpha = \alpha_0 + k\beta_0}$$



From Twiss to Transport Matrix

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$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta_0} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & \frac{1}{\sqrt{\beta_0}} \end{pmatrix} \begin{pmatrix} \sin(\phi_0) \\ \cos(\phi_0) \end{pmatrix}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \sin(\psi(s)+\phi_0) \\ \cos(\psi(s)+\phi_0) \end{pmatrix}$$

$$= \sqrt{2J} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \sin\phi_0 \\ \cos\phi_0 \end{pmatrix}$$

$$\underline{M}(s) = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\psi(s) & \sin\psi(s) \\ -\sin\psi(s) & \cos\psi(s) \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ \frac{\alpha_0}{\sqrt{\beta_0}} & \sqrt{\beta_0} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}}[\cos\psi + \alpha_0 \sin\psi] & \sqrt{\beta_0 \beta} \sin\psi \\ \sqrt{\frac{1}{\beta_0 \beta}}[(\alpha_0 - \alpha)\cos\psi - (1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi - \alpha \sin\psi] \end{pmatrix}$$

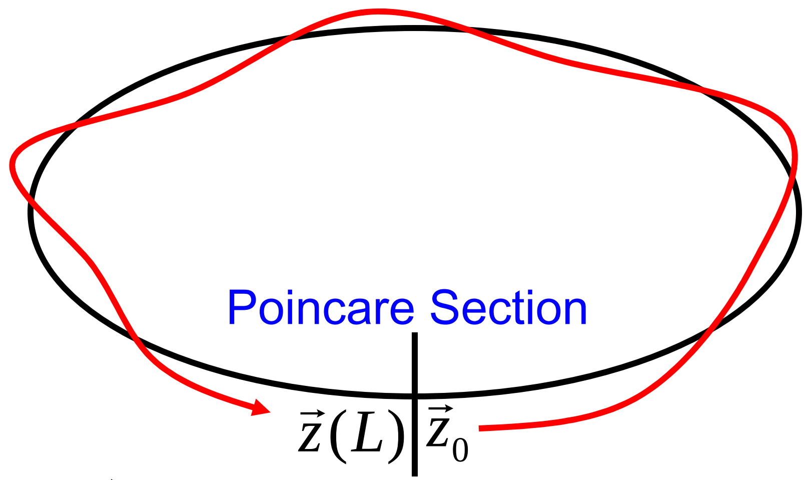
The One Turn Matrix for a Ring

$$\vec{z}(s) = \underline{M}(s,0)\vec{z}(0)$$

$$\vec{z}(L) = \underline{M}(L,0)\vec{z}(0)$$

$$\vec{z}(s+L) = \underline{M}_0(s)\vec{z}(s) \quad , \quad \underline{M}_0 = \underline{M}(s+L,s)$$

$$\vec{z}(s+nL) = \underline{M}_0^n(s)\vec{z}(s)$$

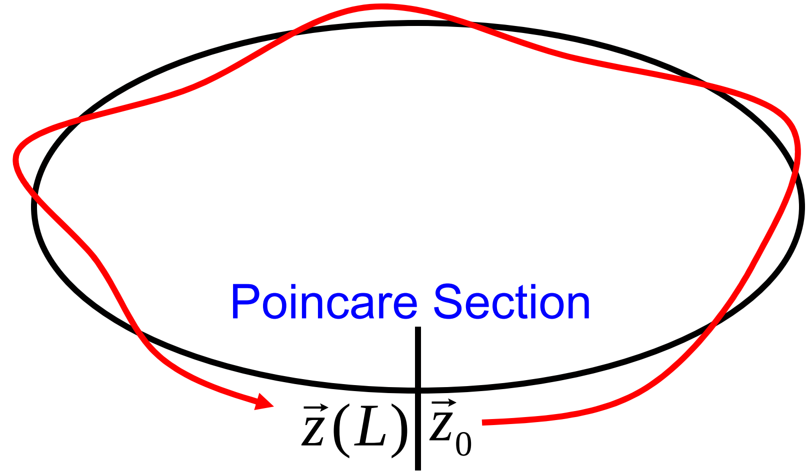


The Periodic Beta Function

If the particle distribution in a ring is stable, it is periodic from turn to turn.

$$\rho(x,x',s+L)=\rho(x,x',s)$$

To be matched to such a beam, the Twiss parameters α, β, γ must be the same after every turn.



$$\underline{M}(s,0)=\begin{pmatrix}\sqrt{\frac{\beta}{\beta_0}}[\cos\psi+\alpha_0\sin\psi] & \sqrt{\beta_0\beta}\sin\psi\\ \sqrt{\frac{1}{\beta_0\beta}}[(\alpha_0-\alpha)\cos\psi-(1+\alpha_0\alpha)\sin\psi] & \sqrt{\frac{\beta_0}{\beta}}[\cos\psi-\alpha\sin\psi]\end{pmatrix}$$

$$\underline{M}_0(s)=\begin{pmatrix}\cos\mu+\alpha\sin\mu & \beta\sin\mu\\ -\gamma\sin\mu & \cos\mu-\alpha\sin\mu\end{pmatrix}=\cos\mu+\begin{pmatrix}\alpha & \beta\\ -\gamma & -\alpha\end{pmatrix}\sin\mu$$

$$\mu=\psi(s+L)-\psi(s)$$

One Turn Matrix to Periodic Twiss

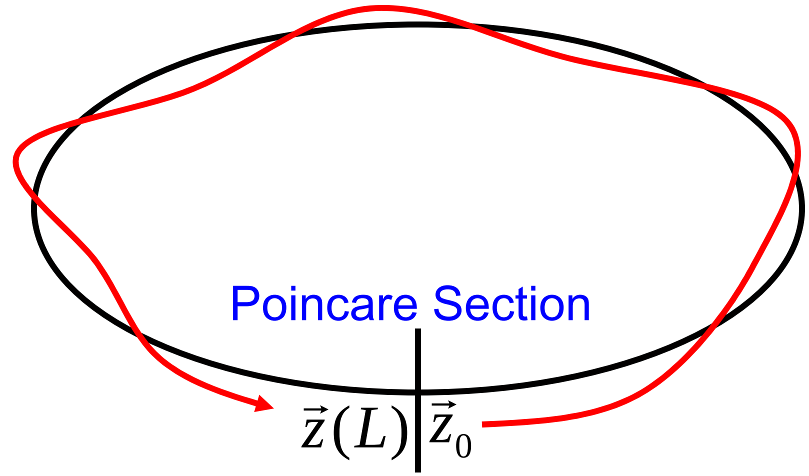
The periodic Twiss parameters are the solution of a nonlinear differential equation with periodic boundary conditions:

$$\beta' = -2\alpha \qquad \text{with} \quad \beta(L) = \beta(0)$$

$$\alpha' = k\beta - \tfrac{1+\alpha^2}{\beta} \qquad \text{with} \quad \alpha(L) = \alpha(0)$$

$$\mu = \int_0^L \tfrac{1}{\beta(\hat{s})} d\hat{s}$$

Note: $\beta(s) > 0$



$$\underline{M}_0(s) = \cos\mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin\mu$$

$$\cos\mu = \tfrac{1}{2}\mathrm{Tr}[\underline{M}_0(s)]$$

$$\beta = \underline{M}_{0,12} \tfrac{1}{\sin\mu}$$

$$\alpha = (\underline{M}_{0,11} - \underline{M}_{0,22}) \tfrac{1}{2\sin\mu}$$

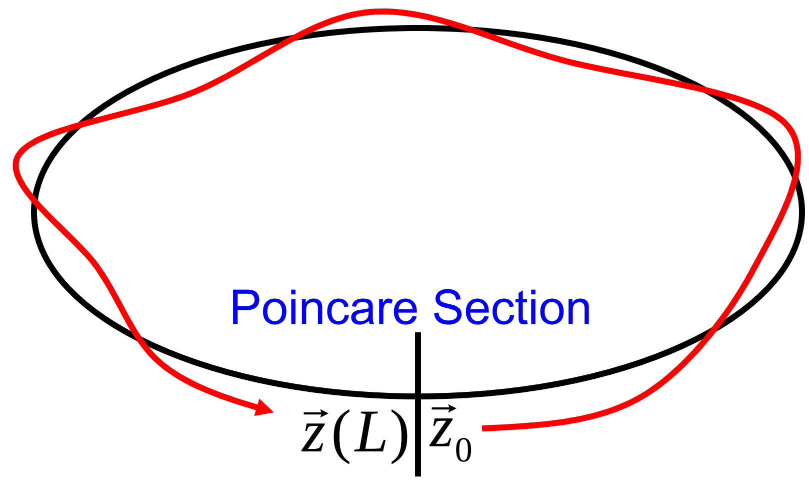
$$\gamma = \tfrac{1+\alpha^2}{\beta}$$

Stable beam motion and thus a periodic beta function can only exist when Tr[$\underline{M}$]<2.

The Tune

The betatron phase advance per turn devided by 2π is called the TUNE.

$$\mu = 2\pi\nu = \psi(s+L) - \psi(s)$$

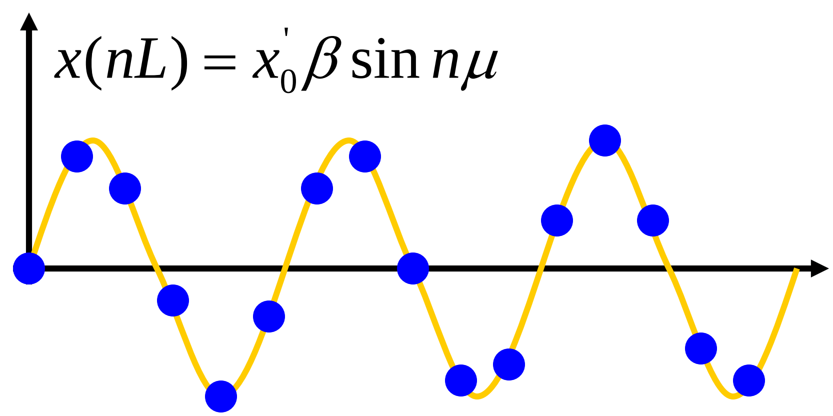


It is a property of the ring and does not depend on the azimuth s.

$$\underline{M}_0(s) = \cos\mu + \begin{pmatrix} -\alpha(s) & \beta(s) \\ \gamma(s) & \alpha(s) \end{pmatrix} \sin\mu$$

$$2\cos\underline{\mu(s)} = \mathrm{Tr}[\underline{M}_0(s)] = \mathrm{Tr}[\underline{M}(s,0)\underline{M}_0(0)\underline{M}_0^{-1}(s,0)]$$

$$= \mathrm{Tr}[\underline{M}_0(0)] = 2\cos\underline{\mu(0)}$$



$$\underline{M}_0^n = \cos n\mu + \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \sin n\mu$$

The Closed Orbit

$$x' = a$$

$$a' = -(\kappa^2 + k)x + \Delta f$$

The extra force can for example come from an erroneous dipole field or from a correction coil: $\Delta f = \frac{q}{p}\Delta B_y = \Delta\kappa$

Variation of constants: $\vec{z} = \underline{M}\vec{z}_0 + \Delta\vec{z}$ with $\Delta\vec{z} = \int_0^s \underline{M}(s-\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$

For the periodic or closed orbit: $\vec{z}_{co} = \underline{M}_0\vec{z}_{co} + \underline{M}_0\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$

$$\vec{z}_{co} = [\underline{M}_0^{-1} - \underline{1}]^{-1}\int_0^L \underline{M}^{-1}(\hat{s})\begin{pmatrix}0\\ \Delta\kappa(\hat{s})\end{pmatrix}d\hat{s}$$

$$= \frac{1}{2-2\cos\mu}[(\cos\mu - 1)\underline{1} + \sin\mu\,\underline{\beta}]\int_0^L \begin{pmatrix}-\sqrt{\beta\hat{\beta}}\sin\hat{\psi}\\ \sqrt{\frac{\hat{\beta}}{\beta}}[\cos\hat{\psi} + \alpha\sin\hat{\psi}]\end{pmatrix}\Delta\kappa(\hat{s})d\hat{s}$$

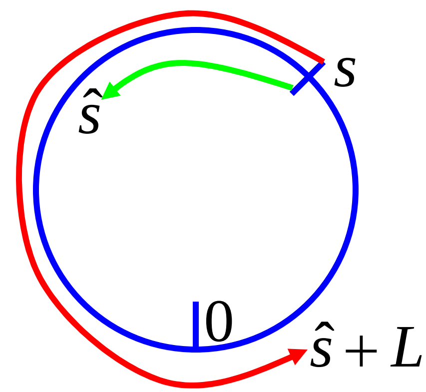
Closed Orbit Integral

$$x_{co}(0) = \frac{1}{2-2\cos\mu}\int_0^L \Delta\hat{\kappa}\sqrt{\beta\hat{\beta}}[(1-\cos\mu)\sin\hat{\psi}+\sin\mu\cos\hat{\psi}]d\hat{s}$$

$$= \frac{1}{4\sin^2\frac{\mu}{2}}\int_0^L \Delta\hat{\kappa}\sqrt{\beta\hat{\beta}}\,2\sin\tfrac{\mu}{2}[\sin\tfrac{\mu}{2}\sin\hat{\psi}+\cos\tfrac{\mu}{2}\cos\hat{\psi}]d\hat{s}$$

$$= \frac{\sqrt{\beta(0)}}{2\sin\frac{\mu}{2}}\int_0^L \Delta\kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(\psi(\hat{s})-\tfrac{\mu}{2})d\hat{s}$$

$$\cos(\int_s^{\hat{s}\{+L\}}\tfrac{1}{\beta}d\hat{s}-\tfrac{\mu}{2}) = \cos(\hat{\psi}-\psi\{+\mu\}-\tfrac{\mu}{2}) = \cos(|\hat{\psi}-\psi|-\tfrac{\mu}{2})$$



The {…} applies when $\hat{s}$ is smaller than s
and therefore $\hat{\psi}$ is smaller than ψ .

$$x_{co}(s) = \frac{\sqrt{\beta(s)}}{2\sin\frac{\mu}{2}}\oint \Delta\kappa(\hat{s})\sqrt{\beta(\hat{s})}\cos(|\psi(\hat{s})-\psi(s)|-\tfrac{\mu}{2})d\hat{s}$$

$$= \sum_k \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi_k-\psi|-\tfrac{\mu}{2})$$

Orbit from One Kick

$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}} \cos(|\psi_k - \psi| - \tfrac{\mu}{2})$$

For $\psi > \psi_k$ this is a free betatron oscillation

$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}} \cos(\psi - \psi_k - \tfrac{\mu}{2})$$

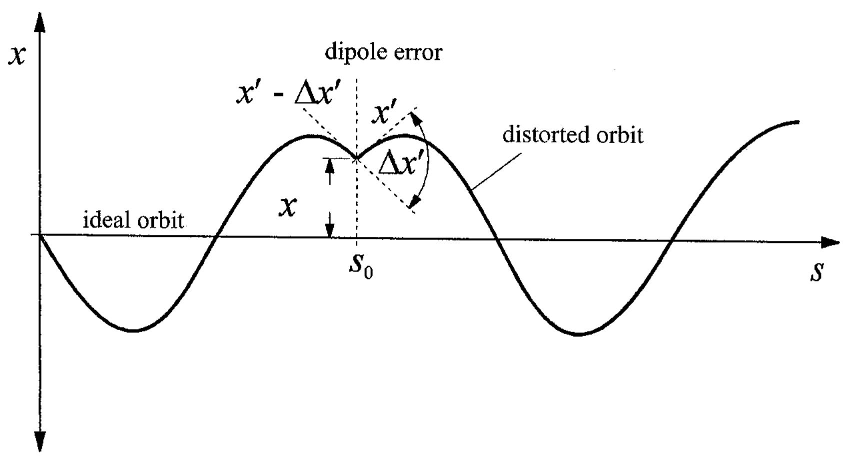
$$= \sqrt{2J\beta(s)}\,\sin(\psi + \phi_0)$$

$$J = \Delta\vartheta_k^2 \frac{\beta_k}{8\sin^2\frac{\mu}{2}}\,, \quad \phi_0 = \tfrac{\pi}{2} - \psi_k - \tfrac{\mu}{2}$$

For $\psi \leq \psi_k$ this is a free betatron oscillation

$$x_{co}(s) = \Delta\vartheta_k \frac{\sqrt{\beta(s)\beta_k}}{2\sin\frac{\mu}{2}} \cos(\psi - \psi_k + \tfrac{\mu}{2})$$

$$J = \Delta\vartheta_k^2 \frac{\beta_k}{8\sin^2\frac{\mu}{2}}\,, \quad \phi_0 = \tfrac{\pi}{2} - \psi_k + \tfrac{\mu}{2}$$



The oscillation amplitude J diverges when the tune ν is close to an integer.

Closed Orbit Correction

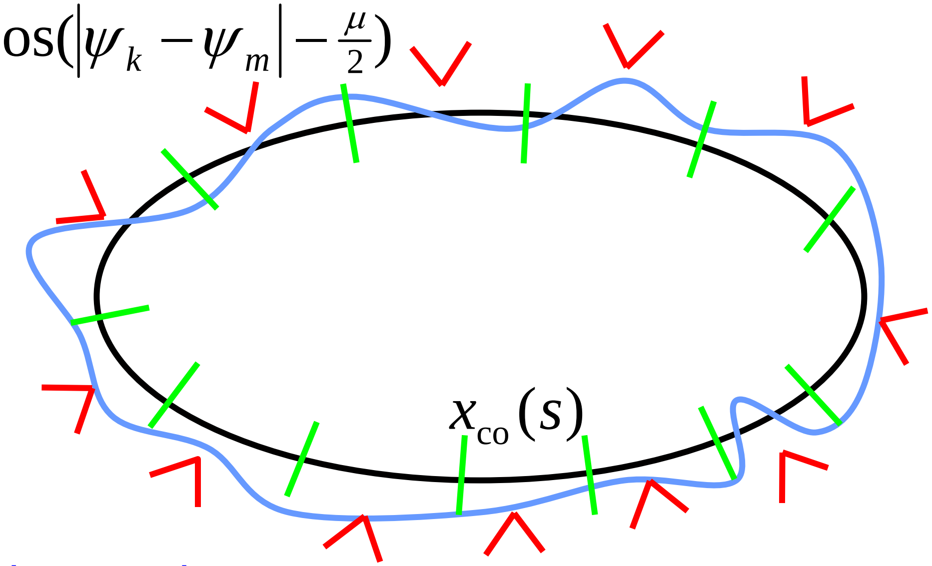
When the closed orbit $x_{co}^{old}(s_m)$ is measured at beam position monitors (BPMs, index m) and is influenced by corrector magnets (index k), then the monitor readings before and after changing the kick angles created in the correctors by $\Delta\vartheta_k$ are related by

$$x_{co}^{new}(s_m) = x_{co}^{old}(s_m) + \sum_k \Delta\vartheta_k \frac{\sqrt{\beta_m\beta_k}}{2\sin\frac{\mu}{2}}\cos(|\psi_k - \psi_m| - \tfrac{\mu}{2})$$

$$= x_{co}^{old}(s_m) + \sum_k O_{mk}\Delta\vartheta_k$$

$$\vec{x}_{co}^{new} = \vec{x}_{co}^{old} + \underline{O}\Delta\vec{\vartheta}$$

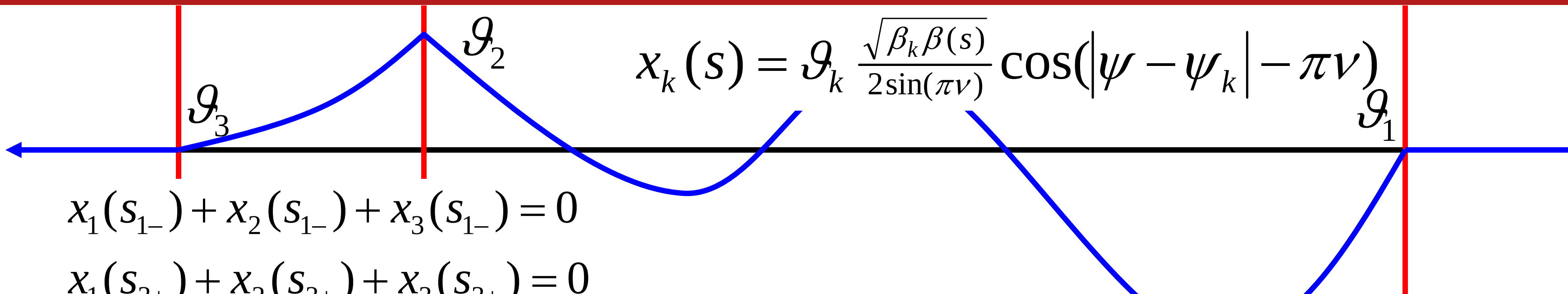
$$\Delta\vec{\vartheta} = -\underline{O}^{-1}\vec{x}_{co}^{old} \quad \Rightarrow \quad \vec{x}_{co}^{new} = 0$$



It is often better not to try to correct the closed orbit at the the BPMs to zero in this way since

1. computation of the inverse can be numerically unstable, so that small errors in the old closed orbit measurement lead to a large error in the corrector coil settings.
2. A zero orbit at all BPMs can be a bad orbit inbetween BPMs

Closed Orbit Bumps



$$x_k(s) = \vartheta_k \frac{\sqrt{\beta_k \beta(s)}}{2\sin(\pi\nu)} \cos(|\psi - \psi_k| - \pi\nu)$$

$$x_1(s_{1-}) + x_2(s_{1-}) + x_3(s_{1-}) = 0$$

$$x_1(s_{3+}) + x_2(s_{3+}) + x_3(s_{3+}) = 0$$

$$\frac{\vartheta_1}{\vartheta_2}\sqrt{\beta_1}\cos(\pi\nu) + \frac{\vartheta_3}{\vartheta_2}\sqrt{\beta_3}\cos(|\psi_3 - \psi_1| - \pi\nu) = -\sqrt{\beta_2}\cos(|\psi_2 - \psi_1| - \pi\nu)$$

$$\frac{\vartheta_1}{\vartheta_2}\sqrt{\beta_1}\cos(|\psi_1 - \psi_3| - \pi\nu) + \frac{\vartheta_3}{\vartheta_2}\sqrt{\beta_3}\cos(\pi\nu) = -\sqrt{\beta_2}\cos(|\psi_2 - \psi_3| - \pi\nu)$$

$$\begin{pmatrix} \frac{\vartheta_1}{\vartheta_2} \\ \frac{\vartheta_3}{\vartheta_2} \end{pmatrix} = \frac{-\sqrt{\beta_2}}{N} \begin{pmatrix} \sqrt{\frac{1}{\beta_1}}\cos(\pi\nu) & -\sqrt{\frac{1}{\beta_1}}\cos(\psi_{31} - \pi\nu) \\ -\sqrt{\frac{1}{\beta_3}}\cos(\psi_{31} - \pi\nu) & \sqrt{\frac{1}{\beta_3}}\cos(\pi\nu) \end{pmatrix} \begin{pmatrix} \cos(\psi_{21} - \pi\nu) \\ \cos(\psi_{32} - \pi\nu) \end{pmatrix}$$

$$N = \cos^2(\pi\nu) - \cos^2(\psi_{31} - \pi\nu) = \sin(\psi_{31} - 2\pi\nu)\sin\psi_{31}$$

$$\begin{pmatrix} \frac{\vartheta_1}{\vartheta_2} \\ \frac{\vartheta_3}{\vartheta_2} \end{pmatrix} = \frac{-1}{N} \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\sin(\psi_{31} - 2\pi\nu)\sin\psi_{32} \\ \sqrt{\frac{\beta_2}{\beta_3}}\sin(\psi_{31} - 2\pi\nu)\sin\psi_{21} \end{pmatrix} = \frac{-1}{\sin\psi_{31}} \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\sin\psi_{32} \\ \sqrt{\frac{\beta_2}{\beta_3}}\sin\psi_{21} \end{pmatrix}$$

$$\vartheta_1 : \vartheta_2 : \vartheta_3 = \beta_1^{-\frac{1}{2}}\sin\psi_{32} : -\beta_2^{-\frac{1}{2}}\sin\psi_{31} : \beta_3^{-\frac{1}{2}}\sin\psi_{21}$$