A Quick Review of Complex Numbers

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The set of complex numbers, commonly symbolized by $\mathbb{C}$, has many applications in physics. Its appearance occured naturally in the 16th century, when mathematicians wanted to express all the roots of polynomials. Integer numbers ($\mathbb{Z}$) could solve equations such as $x+1=0$, rational numbers ($\mathbb{Q}$) could solve equations such as $2x-1=0$, and real numbers ($\mathbb{R}$), equations as $x^2-2=0$. Similarly, complex numbers yield the solutions of equations of the form $x^2+1=0$. The set of complex numbers is now sufficient to express all roots of any polynomial.

First, we give formal definitions and we show some of the most important properties of complex numbers. Then, we present a short list of the many possible applications related to complex numbers.

1 Definition

A complex number is an ordered pair of two real numbers, x and y, written

$$z = x + iy, \tag{1}$$

where $i=\sqrt{-1}$. x is called the *real part* and y, the *imaginary part* of z. This is written as $\Re\{z\}=x$ and $\Im\{z\}=y$.

You know already that a pair of real numbers specifes the position of a point on a 2-dimensional cartesian plane. The set of complex numbers defines the *complex plane* the same way. It is therefore natural to represent a complex number in polar coordinates:

$$z = r(\cos\theta + i\sin\theta), \tag{2}$$

with the usual relations between (x,y) and (r,θ).
Exercise: Write down these relations. You shouldn't need to think more than 2 or 3 seconds!

Here, r is called the *modulus* (or, sometimes, the absolute value) and θ is the *phase* of z. We often write $r \equiv |z|$. Using the Taylor expansion of cos() and sin() functions, this can be written in the very elegant, so-called *complex exponential* form:

$$z = re^{i\theta}. \tag{3}$$

Exercise: Assuming that sin(), cos() and exp() are well-defined in the whole complex plane, prove that (2) is equivalent to (3).
Exercise: Using the equality of (2) and (3), express $cos(x)$ and $sin(x)$ as a combination of complex exponentials.

2 Basic Operations

The addition of two complex numbers is totally equivalent to the usual vectorial componant-by-componant addition:

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2). \tag{4}$$

As you can guess, additions are most easily done in the $x + iy$ notation.
Exercise: What is the addition of two complex numbers in the complex exponential notation?

The multiplication of two complex numbers is also the usual, distributive, commutative, multiplication that we used in $\mathbb{R}$. You only have to remember that $i^2 = -1$!
Exercise: What is $z_1 z_2$ in the notation (1)?

Here, the complex exponential notation is really more effective:

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}. \tag{5}$$

Exercise: Is it true that $\Re\{z_1 + z_2\} = \Re\{z_1\} + \Re\{z_2\}$? And what about $\Re\{z_1 z_2\} = \Re\{z_1\}\Re\{z_2\}$?
Exercise: Show that $z^2 \neq |z|^2$ in general. This confusion is one common source of miscalculations.

It should be clear for you that the result of the multiplication of two complex numbers is a complex number which modulus is the product of the modulus of the two numbers, and which phase is the sum of the two phases.
Exercise: What complex number represents a pure rotation of an angle ϕ?

3 Complex Conjugation

The complex conjugate of a complex number z is noted z^* (mathematicians also often use $\bar{z}$) and is defined by:

$$z^* = (x + iy)^* = x - iy. \tag{6}$$

Hence, complex conjugation correspond to a transformation of a number into its symmetric part with respect to the real axis. In complex exponential notation, it is:

$$z^* = (re^{i\theta})^* = re^{-i\theta}.$$

A good trick is to remember that taking the complex conjugate is equivalent to replacing all i's by $-i$'s.
Exercise: Show that $(z_1 z_2)^* = z_1^* z_2^*$.
Exercise: What do you conclude of a number such that $z^* = z$? And if $z^* = -z$?
Exercise: Express $|z|^2$ as a function of z and z^*. Isn't it nice?
Exercise: Show that $|z_1 z_2| = |z_1||z_2|$.

Now, a few applications.

4 Waves and Oscillations

Recall the differential equation of a harmonic oscillator:

$$\frac{d^2\psi}{dt^2} + \omega^2 \psi = 0. \tag{7}$$

Since this is a linear ODE, its solution can be written $\psi(t) = Ae^{\lambda t}$. Substituting, you get the condition $\lambda^2 = -\omega^2$, that is, $\lambda = \pm i\omega$. The general solution is therefore of the form:

$$\psi(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t}. \tag{8}$$

Where $A_1, A_2 \in \mathbb{C}$. Of course, if you want this equation to represent a physical system, you have to specify properly A_1 and A_2.

The wave equation,

$$\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}, \tag{9}$$

is also linear and can be solved the same way (using the method of separation of variables): the solutions can be expressed as plane waves, of the form $Ae^{i(\pm kz \pm \omega t)}$. The informations on the physical system should suffice to specify the value of the complex coefficients, and may impose constraints on the values of ω and k.

Finally, you will soon encounter the famous Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V\Psi, \tag{10}$$

which is also linear when V is indenpendant of time and position. Once again, the solutions will be easily expressed in the form of plane waves (complex exponentials).

5 Fourier Transforms

Another field of application of complex numbers, relevant to quantum mechanics, is Fourier transforms. Since you might have never seen the theory, we won't present it in detail. At some point, you will certainly have to calculate integrals such as

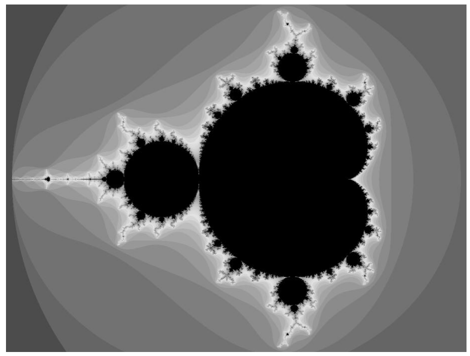
$$g(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx. \tag{11}$$

Here again, it is essential to know how to manipulate complex exponentials.

6 Other Application

Even outside Quantum Mechanics, complex numbers are everywhere.

In physics, for instance, you are likely to meet them in electromagnetism (when dealing with oscillating currents). The mathematics field which deals with the complex plane is called Complex Analysis. If you learn this subject, you will learn, among other things, amazing properties of some functions defined on the complex plane. You will also see some nice theorems about integration of functions along a path in $\mathbb{C}$.



Complex numbers can even make stunning and beautiful images. The Mandelbrot set is the classic example of such fractals.

Technical notice: this set is realized quite easily by iteration. For each point c on the complex plane, set $z_0 = 0$, then compute $z_1 = c$, $z_2 = z_1^2 + c$, $z_3 = z_2^2 + c$, ... $z_{n+1} = z_n^2 + c$. You make the image by relating a color to the "rate" at which $|z_n|$ goes to infinity.