P6572 HW #6
Due October 7, 2011

1. **Sakurai, p. 242, Problem 2**
   Consider the 2X2 matrix defined by
   $$U = \frac{a_0 + i\sigma \cdot \mathbf{a}}{a_0 - i\sigma \cdot \mathbf{a}},$$
   where a_0 is a real number and $\mathbf{a}$ is a three-dimensional vector with real components.
   (a) Prove that U is unitary and unimodular.
   (b) In general, a 2X2 matrix represents a rotation in three dimensions. Find the axis and angle of rotation appropriate for U in terms of a_0, a_1, a_2, and a_3.

2. **Generators**
   (a) Show that in any representation where J_x and J_z are real matrices (therefore symmetrical), J_y is a pure imaginary matrix (therefore antisymmetrical).
   (b) Show that if any operator commutes with *two* components of an angular momentum vector, it commutes with the third.
   (c) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three unit vectors forming a right-handed Cartesian system. Show that the infinitesimal rotation
   $$\hat{R} \equiv R_v^{-1}(\epsilon) R_u^{-1}(\epsilon) R_v(\epsilon) R_u(\epsilon)$$
   differs from $R_w(-\epsilon^2)$ only by terms of higher order than ϵ^2.

3. **Sakurai, p. 242, Problem 3**
   The spin dependent Hamiltonian of an electron-positron system in the presence of a uniform magnetic field in the z-direction can be written as
   $$H = A\mathbf{S}^{(\mathbf{e}^-)} \cdot \mathbf{S}^{(\mathbf{e}^+)} + \left(\frac{eB}{mc}\right)(S_z^{(e^-)} - S_z^{(e^+)}).$$
   Suppose the spin function of the system is given by $\chi_+^{(e^-)}\chi_-^{(e^+)}$.
   (a) Is this an eigenfunction of H in the limit $A \to 0, eB/mc \neq 0$? If it is, what is the energy eigenvalue? If it is not, what is the expectation value of H?
   (b) Same problem when $eB/mc \to 0, A \neq 0$.

4. **Sakurai, p. 242, Problem 5**
   Let the Hamiltonian of a rigid body be

$$H = \frac{1}{2}\left(\frac{K_1^2}{I_1} + \frac{K_2^2}{I_2} + \frac{K_3^2}{I_3}\right)$$

   where $\mathbf{K}$ is the angular momentum in the body frame. From this expression obtain the Heisenberg equation of miotion for $\mathbf{K}$ and then find Euler's equation of motion in the correspondence limit.

5. **Sakurai, p. 243, problem 8**

   Consider a sequence of Euler rotations represented by

$$\begin{aligned} \mathcal{D}^{(1/2)}(\alpha,\beta,\gamma) &= \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right) \\ &= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix} \end{aligned}$$

   Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation abut some axis by an angle θ. Find θ.

6. **Unstable States**

   We can write the time evolution of a state as $\mid \psi(t)\rangle = \exp(-iEt/\hbar)\mid \psi(0)\rangle$. If we start in that state $\mid \psi(0)\rangle$, the probability of remaining in the state remains constant: $P = |\langle \psi(t) \mid \psi(t)\rangle|^2 = |\langle \psi(0) \mid \psi(0)\rangle|^2$. If we make the substitution $E \to E - i\frac{\hbar\gamma}{2}$, then the probability of remaining in the state is no longer constant: $P = |\langle \psi(t) \mid \psi(t)\rangle|^2 = \exp(-\gamma t)$. A "complex energy" means that the hamiltonian is no longer hermitian.

   Now consider a state $\mid \phi_2\rangle$ which is stable and a state $\mid \phi_1\rangle$ that decays with lifetime $\tau_1 = 1/\gamma_1$. The Hamiltonian is:

$$H_0 = \begin{pmatrix} E_1' & 0 \\ 0 & E_2' \end{pmatrix} = \begin{pmatrix} E_1 - i\frac{\hbar\gamma_1}{2} & 0 \\ 0 & E_2 \end{pmatrix}$$

   If we now turn on a coupling between states $\mid \phi_1\rangle$ and $\mid \phi_2\rangle$,

$$H = H_0 + W = \begin{pmatrix} E_1 - i\frac{\hbar\gamma_1}{2} & W_{12} \\ W_{21} & E_2 \end{pmatrix}$$

   where $W_{12} = W_{21}^*$.

(a) Solve for the new eigen energies and show that in the limit of weak coupling $|W_{12}| \ll \sqrt{(E_1 - E_2)^2 + \frac{\hbar^2\gamma^2}{4}}$, the new eigenenergies are:

$$\begin{aligned}\epsilon_1' &= E_1 - i\frac{\hbar\gamma_1}{2} + \frac{|W_{12}|^2}{E_1 - E_2 - i\hbar\gamma_1/2} \\ \epsilon_2' &= E_2 + \frac{|W_{12}|^2}{E_2 - E_1 + i\hbar\gamma_1/2}\end{aligned}$$

The energies of the eigenstates in the presence of the coupling are the real parts of ϵ_1' and ϵ_2'; the lifetimes are inversely proportional to their imaginary parts. In particular, we see that ϵ_1' and ϵ_2' are both complex when $|W_{12}|$ is not zero. In the presence of the coupling there is no longer any stable state.

(b) Re-write ϵ_2' as $\epsilon_2' = \Delta_2 - i\frac{\hbar\Gamma_2}{2}$ and calculate expressions for Δ_2 and Γ_2.

(c) Let $E_1 = E_2$ for simplicity. Solve for the probability of finding the system in the state $|\,\phi_1\rangle$ when $|W_{12} > \frac{\hbar\gamma_1}{4}$ and the system is initially in the state $|\,\phi_2\rangle$.

(d) Still under the condition $E_1 = E_2$, solve for the probability of finding the system in the state $|\,\phi_1\rangle$ when $|W_{12}| < \frac{\hbar\gamma_1}{4}$ and the system is initially in the state $|\,\phi_2\rangle$.

(e) Offer a physical interpretation of the results you have just derived.

7. **Antiparticles and antigravity**

We now want to consider a practical application of the previous problem.

The K_0 meson and its antiparticle $\bar{K}_0$ can be produced in a reaction $\pi^- + p \to K_0 + \bar{K}_0$+other stuff. However, the decay modes of the K_0 meson are given in terms of linear superposition of states:

$$\begin{aligned}K_L &= \frac{1}{\sqrt{2}}\left(|\,K_0\rangle + |\,\bar{K}_0\rangle\right) \\ K_S &= \frac{1}{\sqrt{2}}\left(|\,K_0\rangle - |\,\bar{K}_0\rangle\right)\end{aligned}$$

K_S decays into 2 pions in approximately 10^{-10} seconds, while K_L decays (into 3 pions) with a life time 600 times longer. (Actually K_L has been seen to decay into 2 pions with a rate 10^{-3} of the 3π decay rate due to CP violation.)

People have suggested that antiparticles fall up and want to make anti-hydrogen to test this hypothesis. We will analyze this conjecture with regard to the $K_0 - \bar{K}_0$ system. Assume the K_L is in the gravitational potential of the earth $V = -GMm/R$, where M is the mass of the earth and R is the radius of the earth.

(a) If the gravitational mass of $\bar{K}_0$ has the opposite sign of the K_0, derive an expression for how long it would take for K_L to decay into 2 π's via an oscillation into K_S. Given the known decay rate into 2 π's , what is the upper limit for the gravitational mass difference of K_0 and $\bar{K}_0$?

(b) Give a numerical answer to part (a). It will help to know that the mass of the K meson is approximately $500\mathrm{MeV/c^2}$.