Problem 1

The propagator

$$K(x,t;x'\,t') \;\equiv\; \langle x,t|x',t'\rangle$$

can be rewritten in terms of the energy eigenstates ψ_n (with energy E_n) as follows:

$$K(x,t;x',t') \;=\; \sum_n \psi_n(x)\,\psi_n(x')^\star\,e^{-i\,E_n\,(t-t')/\hbar}$$

Suppose we perform a path integral to obtain the SHO propagator:

$$K(x,t;x',0) \;=\; \sqrt{\frac{m\,\omega}{2\pi i\hbar\,\sin\omega t}}\;\exp\!\left[\frac{im\omega}{2\hbar\sin\omega t}\left[\left(x^2+(x')^2\right)\cos\omega t - 2\,x\,x'\right]\right]$$

where we set $t'=0$ WLOG. Now we want to extract information about the eigenstates and their energies.

a)

We set $x=x'=0$ to obtain:

$$\sum_n |\psi_n(0)|^2\,e^{-i\,E_n\,t/\hbar} \;=\; \sqrt{\frac{m\,\omega}{2\pi i\hbar\,\sin\omega t}}$$

We perform Fourier analysis on the RHS to determine the eigenenergies. We need a prescription for going around the branch cut singularity of $\sqrt{u}$ located at $u=0$. The correct prescription is to add a small negative imaginary piece to the time:

$$t \;\rightarrow\; t - i\,\varepsilon/\omega$$

where ε is a infinitesimal dimensionless regulator. Thus, $\sin\omega t \rightarrow \sin\left(\omega\,t - i\varepsilon\right) = \cosh\varepsilon\,\sin\omega t - i\sinh\varepsilon\,\cos\omega t \simeq \sin\omega t - i\,\varepsilon\cos\omega t$, and we find:

$$\sum_n |\psi_n(0)|^2\,e^{-i\,E_n\,t/\hbar-\varepsilon\,E_n/(\hbar\omega)} \;=\; \sqrt{\frac{m\,\omega}{2\pi i\hbar\,\left(\sin\omega t - i\,\varepsilon\cos\omega\,t\right)}}$$

To see that this is the correct prescription, check the case $t=0$:

$$\sum_n |\psi_n(0)|^2\,e^{-\varepsilon\,E_n/(\hbar\omega)} \;=\; \sqrt{\frac{m\,\omega}{2\pi\hbar\,\varepsilon}}$$

The LHS is a convergent, regulator dependent sum, and is therefore a positive real number, as is the RHS.

Having regulated the propagator, we see that $K(0,t;0,0)$ has a period $4\pi/\omega$, rather than the naive expectation $2\pi/\omega$. This is because $\frac{1}{\varepsilon\cos\omega t + i\sin\omega t}$ crosses the $\sqrt{u}$ branch cut (conventionally along the negative real axis) once each "period" $2\pi/\omega$ (at times $t_{\rm br} = (\pi + 2\pi n)/\omega$), picking up an extra minus sign; thus, $K(0,t;0,0)$ satisfies $K(0,t+2\pi/\omega;0,0) = -\,K(0,t;0,0)$[1].

We write $K(0,t;0,0)$ as a Fourier series:

$$K(0,t;0,0) \;=\; \sum_n c_n\, e^{-in\omega t/2}$$

Thus,

$$c_n \;=\; \frac{\omega}{4\pi}\int_0^{4\pi/\omega} K(0,t;0,0)\, e^{in\omega t/2}\,\mathrm{d}t$$

The antiperiodicity of $K(0,\, t;\, 0,\, 0)$ over the interval $2\pi/\omega$ implies that the c_n vanish for n even. In principle, all we have to do to extract the other coefficients (for n odd) is to perform the above integral. However, this is somewhat difficult to do in practice.

Consider the following shortcut: instead of adding a small regulator term, suppose we add a large negative imaginary part to t:

$$t \;\rightarrow\; t - i\,\alpha/\omega$$

where $\alpha > 0$. We find:

$$\sum_n |\psi_n(0)|^2\, e^{-iE_n t/\hbar - \alpha E_n/(\hbar\omega)} \;=\; \sqrt{\frac{m\,\omega}{\pi\hbar\,(\,e^{\alpha + i\omega t} - e^{-\alpha - i\omega t})}}$$

Suppose we take the limit $\alpha \rightarrow \infty$. The RHS vanishes, whereas the LHS only vanishes if all the E_n are positive. This must therefore be the case. Now take α to be large but finite, and expand the RHS in a power series in $e^{-\alpha}$ (we set $t = 0$ for simplicity; it's easy to see that the $e^{i\omega t}$'s always line up when the e^{α}'s do):

$$\begin{aligned}\sum_n |\psi_n(0)|^2\, e^{-\alpha E_n/(\hbar\omega)} \;&=\; e^{-\alpha/2}\sqrt{\frac{m\omega}{\pi\hbar}}\,\left(1 - e^{-2\alpha}\right)^{-1/2} \\ &\simeq\; e^{-\alpha/2}\sqrt{\frac{m\omega}{\pi\hbar}}\left[1 + \frac{1}{2}\,e^{-2\alpha} + \frac{3}{8}\,e^{-4\alpha} + \ldots\right]\end{aligned}$$

We read off the eigenenergies:

$$E_n \;=\; \hbar\,\omega/2,\; 5\hbar\,\omega/2,\; 9\hbar\,\omega/2, \ldots$$

and the coefficients:

$$\begin{aligned}|\psi_0(0)|^2 \;&=\; \sqrt{\frac{m\omega}{\pi\hbar}} \\ |\psi_2(0)|^2 \;&=\; \frac{1}{2}\sqrt{\frac{m\omega}{\pi\hbar}} \\ |\psi_4(0)|^2 \;&=\; \frac{3}{8}\sqrt{\frac{m\omega}{\pi\hbar}} \\ &\;\;\vdots\end{aligned}$$

1. This property is obvious when we obtain the propagator from the energy eigenstates, as the eigenenergies are $E = (n + 1/2)\,\hbar\omega$, where the $+1/2$ contributes the crucial minus sign.

Looking at Sakurai eqn. 2.6.17, We see that these coefficients should take the general form:

$$|\psi_n(0)|^2 \;=\; \frac{1}{2^n\, n!}\,[H_n(0)]^2\,\sqrt{\frac{m\omega}{\pi\hbar}}$$

It's easy to verify that the first few work out as expected.

The odd energy levels do not appear since their wavefunctions are odd functions of x, so that $\psi_1(0) = \psi_3(0) = \ldots = 0$.

Why is this procedure valid? Suppose the regulated sum $\sum_n |\psi_n(0)|^2\, e^{-i E_n t/\hbar - \varepsilon E_n/(\hbar\omega)}$ is convergent for ε infinitesimal. Increasing ε can only make it more convergent. Thus, taking $\alpha \to \infty$, we obtain the analytic continuation of the propagator to $t = -i\alpha/\omega$ for α large. Since the RHS is analytic, we must obtain the same result upon analytically continuing it to $t = -i\alpha/\omega$, allowing us to match coefficients in a regime where the expansion in $e^{-\alpha}$ is valid.

b)

We generalize the procedure of the previous section to the case $x = x' \neq 0$. The propagator becomes:

$$\begin{aligned} K(x,t;x,0) &= \sqrt{\frac{m\,\omega}{2\pi i\hbar\,\sin\omega t}}\,\exp\left[-\frac{m\,\omega\,x^2}{\hbar}\left(\frac{\cos\omega t - 1}{i\sin\omega t}\right)\right] \\ &= \sqrt{\frac{m\,\omega}{\pi\,\hbar\,(e^{i\omega t} - e^{-i\omega t})}}\,\exp\left[-\frac{m\,\omega\,x^2}{\hbar}\left(\frac{e^{i\omega t} + e^{-i\omega t} - 2}{e^{i\omega t} - e^{-i\omega t}}\right)\right] \\ &= \sqrt{\frac{m\,\omega\,e^{-i\omega t}}{\pi\hbar\,(1 - e^{-2i\omega t})}}\,\exp\left[-\frac{m\,\omega\,x^2}{\hbar}\left(\frac{1 - 2e^{-i\omega t} + e^{-2i\omega t}}{1 - e^{-2i\omega t}}\right)\right] \end{aligned}$$

We perform the same procedure as above, replacing $t = -i\alpha/\omega$ and expanding in powers of $a \equiv e^{-\alpha}$. The term in parentheses inside the exponent becomes:

$$\begin{aligned} \ldots &= \frac{1 - 2a + a^2}{1 - a^2} \\ &= \frac{(1-a)^2}{(1-a)\,(1+a)} \\ &= \frac{1-a}{1+a} \\ &= 1 - \frac{2a}{1+a} \end{aligned}$$

Thus, defining $\gamma^2 \equiv \frac{m\,\omega\,x^2}{\hbar}$, we find:

$$\begin{aligned} \sum_n |\psi_n(x)|^2\, a^{(E_n/\hbar\omega)} &= a^{1/2}\,(1-a^2)^{-1/2}\sqrt{\frac{m\,\omega}{\pi\hbar}}\, e^{-\gamma^2}\exp\left[\frac{2a\gamma^2}{1+a}\right] \\ &= a^{1/2}\sqrt{\frac{m\,\omega}{\pi\hbar}}\, e^{-\gamma^2}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\,a^n\left[\frac{\partial^n}{\partial a^n}\left(\frac{1}{\sqrt{1-a^2}}\exp\left[\frac{2\,a\,\gamma^2}{1+a}\right]\right)\right]_{a=0}\right] \\ &= \sqrt{\frac{m\,\omega}{\pi\hbar}}\left[a^{1/2} + 2\gamma^2\,a^{3/2} + \frac{1}{2}\,(2\gamma^2 - 1)^2\,a^{5/2} + \ldots\right] e^{-\gamma^2} \end{aligned}$$

Thus, we find the spectrum $E_n = (n+1/2)\,\hbar\omega$, and the modulus squared of the corresponding wavefunctions:

$$\begin{aligned}
|\psi_0(x)|^2 &= \sqrt{\frac{m\,\omega}{\pi\,\hbar}}\,\exp\left[-\frac{m\,\omega\,x^2}{\hbar}\right] \\
|\psi_1(x)|^2 &= 2\sqrt{\frac{m\,\omega}{\pi\,\hbar}}\left[\frac{m\,\omega\,x^2}{\hbar}\right]\exp\left[-\frac{m\,\omega\,x^2}{\hbar}\right] \\
|\psi_2(x)|^2 &= \frac{1}{2}\sqrt{\frac{m\,\omega}{\pi\,\hbar}}\left(\frac{2\,m\,\omega\,x^2}{\hbar}-1\right)^2\exp\left[-\frac{m\,\omega\,x^2}{\hbar}\right] \\
&\vdots
\end{aligned}$$

These are easily checked against Sakurai eqn. 2.6.17:

$$|\psi_n(x)|^2 = \frac{1}{2^n\,n!}\sqrt{\frac{m\,\omega}{\pi\,\hbar}}\left[H_n\!\left(\sqrt{\frac{m\omega}{\hbar}}\,x\right)\right]^2\exp\left[-\frac{m\,\omega\,x^2}{\hbar}\right]$$

The next step

If we assume that the wavefunctions are real, then the above procedure is sufficient to recover them from the propagator. Obviously, this is not always a good assumption. To circumvent it, we let x and x' take arbitrary values. Writing this out, we find:

$$K(x,t;x',0) = \sqrt{\frac{m\,\omega}{\pi\hbar\,(e^{i\omega t}-e^{-i\omega t})}}\,\exp\left[-\frac{m\omega}{\hbar\,(e^{i\omega t}-e^{-i\omega t})}\left[\frac{1}{2}\left(x^2+(x')^2\right)\left(e^{i\omega t}+e^{-i\omega t}\right)-2\,x\,x'\right]\right]$$

Replacing $e^{-i\omega t}$ with a, and definining $\gamma^2 \equiv \frac{m\omega x^2}{\hbar}$ and $\gamma'^2 \equiv \frac{m\omega x'^2}{\hbar}$ as above, we obtain:

$$\begin{aligned}
\sum_n \psi_n(x)\,\psi_n(x')^\star\,a^{(E_n/\hbar\omega)} &= \frac{a^{1/2}}{\sqrt{1-a^2}}\sqrt{\frac{m\,\omega}{\pi\,\hbar}}\,\exp\left[-\frac{1}{1-a^2}\left[\frac{1}{2}\left(\gamma^2+(\gamma')^2\right)\left(1+a^2\right)-2\,a\,\gamma\gamma'\right]\right] \\
&= \frac{a^{1/2}}{\sqrt{1-a^2}}\sqrt{\frac{m\,\omega}{\pi\,\hbar}}\,\exp\left[-\frac{\gamma^2+\gamma'^2}{2}\right]\exp\left[\frac{2\,a\,\gamma\gamma'-a^2\left(\gamma^2+\gamma'^2\right)}{1-a^2}\right]
\end{aligned}$$

Expanding the RHS in a power series in a, we find:

$$\sum_n \psi_n(x)\,\psi_n(x')^\star\,a^{(E_n/\hbar\omega)} = \sqrt{\frac{m\,\omega}{\pi\,\hbar}}\left[a^{1/2}+2\,\gamma\,\gamma'\,a^{3/2}+\frac{1}{2}\,(2\,\gamma^2-1)(2\,\gamma'^2-1)+\ldots\right]\exp\left[-\frac{\gamma^2+\gamma'^2}{2}\right]$$

Factoring the answer, we read off the wavefunctions:

$$\begin{aligned}
\psi_0(x) &= \left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\exp\left[-\frac{1}{2}\,\gamma^2\right] \\
\psi_1(x) &= \sqrt{2}\,\gamma\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\exp\left[-\frac{1}{2}\,\gamma^2\right] \\
\psi_2(x) &= \frac{1}{\sqrt{2}}\,(2\,\gamma^2-1)\left(\frac{m\,\omega}{\pi\,\hbar}\right)^{1/4}\exp\left[-\frac{1}{2}\,\gamma^2\right] \\
&\vdots
\end{aligned}$$

where the overall phase is chosen arbitrarily.

Problem 2

We wish to find the momentum space propagator:

$$K(\boldsymbol{p},t;\boldsymbol{p}',t') \;\equiv\; \langle \boldsymbol{p},t|\boldsymbol{p}',t'\rangle$$

where the state $|p,t\rangle$ satisfies:

$$\hat{\boldsymbol{p}}_H(t)\,|\boldsymbol{p},t\rangle \;=\; \boldsymbol{p}\,|\boldsymbol{p},t\rangle$$

and therefore takes the form:

$$\begin{aligned} |\boldsymbol{p},t\rangle &= \mathcal{U}^\dagger(t)\,|\boldsymbol{p}\rangle \\ &= \exp\left(\frac{iHt}{\hbar}\right)|\boldsymbol{p}\rangle \end{aligned}$$

where $|\boldsymbol{p}\rangle \equiv |\boldsymbol{p},0\rangle$ is the Schrödinger picture momentum eigenstate. Thus, for a free particle, with Hamiltonian $H = \frac{|\hat{\boldsymbol{p}}|^2}{2m}$,

$$\begin{aligned} K(\boldsymbol{p},t;\boldsymbol{p}',t') &= \langle \boldsymbol{p}|e^{-iH(t-t')/\hbar}|\boldsymbol{p}'\rangle \\ &= e^{-i\frac{|\boldsymbol{p}|^2}{2m}(t-t')/\hbar}\,\langle \boldsymbol{p}|\boldsymbol{p}'\rangle \\ &= \delta^{(d)}(\boldsymbol{p}-\boldsymbol{p}')\,\exp\left[-i\,\frac{|\boldsymbol{p}|^2(t-t')}{2m\hbar}\right] \end{aligned}$$

where d is the number of dimensions in the problem (typically 1, 2, or 3).

Problem 3

We want to solve the differential equation

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - E\right)G(x,x',E) \;=\; \delta(x-x')$$

where $G(x,x',E) = G(x-x',E)$ is the one-dimensional Green's function.

There are various tricks for obtaining Green's functions. For instance, one can Fourier transform the above equation to find an algebraic equation for $\tilde{G}(p)$, which is easily solved. Contour integration can then be used to obtain $G(x)$.

However, in this case, a brute force approach is tractable. We set $x' = 0$ WLOG to obtain

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - E\right)G(x,E) \;=\; \delta(x)$$

We first write down the homogenous solutions (without the delta-function source), which take the form:

$$G_H(x,E) \;=\; Ae^{ik_Ex} + Be^{-ik_Ex}$$

where $k_E \equiv \sqrt{2mE}/\hbar$. Since the source is localized at $x = 0$, we must have

$$G(x,E) \;=\; \begin{cases} A e^{ik_E x} + B e^{-ik_E x} & x < 0 \\ C e^{ik_E x} + D e^{-ik_E x} & x > 0 \end{cases}$$

for some coefficients A, B, C and D. To determine the relationship between A, B and C, D, we integrate the defining differential equation from $x = -\varepsilon$ to $x = \varepsilon$ for small ε:

$$-\left[\frac{\hbar^2}{2m}\frac{\partial}{\partial x}G(x,E)\right]_{-\varepsilon}^{\varepsilon} - E\int_{-\varepsilon}^{\varepsilon} G(x,E)\,\mathrm{d}x \;=\; 1$$

The contribution from the second term on the LHS is infinitesimal, and can be dropped. We then find

$$\left[\frac{\partial}{\partial x}G(x,E)\right]_{x=\varepsilon} - \left[\frac{\partial}{\partial x}G(x,E)\right]_{x=-\varepsilon} \;=\; -\frac{2m}{\hbar^2}$$

or

$$(ik_E C - ik_E D) - (ik_E A - ik_E B) \;=\; -\frac{2m}{\hbar^2}$$

Continuity of $G(x,E)$ at $x = 0$ further requires

$$A + B \;=\; C + D$$

The solution is

$$C - A \;=\; B - D \;=\; i\,\frac{m}{\hbar^2\,k_E}$$

To fix the remaining ambiguities, we need to specify boundary conditions. For potential scattering problems ($E > 0$), we may want to exclude incoming waves. To do so, we must set $A = D = 0$. Thus,

$$G(x,E) \;=\; i\,\frac{m}{\hbar^2\,k_E}\;e^{ik_E\,|x|}$$

This is also an appropriate choice for bound state problems ($E < 0$), where $k_E = i\kappa_E$ for $\kappa_E \equiv \sqrt{-2mE}/\hbar$, so that

$$G(x,E) \;=\; \frac{m}{\hbar^2\,\kappa_E}\;e^{-\kappa_E\,|x|}$$

and therefore $G \to 0$ as $x \to \pm\infty$.

In either case, for $E \to 0$ we find

$$G(x,E) \;=\; -\frac{m}{\hbar^2}\,|x|$$

after discarding a divergent constant piece and terms which vanish as $E \to 0$. This is easily seen to satisfy the defining equation, since

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}|x| \;=\; \delta(x)$$

which can be verified by integrating both sides.

Other choices of Green's function will differ by the homogenous solutions $G_H(x, E)$ discussed above.

Solution by contour integration

For completeness, we also present the contour integration method. To begin with, we Fourier transform. Define

$$\tilde{G}(p, E) \;\equiv\; \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}x\; e^{-ipx/\hbar}\, G(x, E)$$

We start with the defining differential equation

$$\left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - E\right) G(x, E) \;=\; \delta(x)$$

Multiplying by $\frac{1}{\sqrt{2\pi\hbar}}\, e^{-ipx/\hbar}$ and integrating over x, we eventually find

$$\left(\frac{p^2}{2m} - E\right) \tilde{G}(p, E) \;=\; \frac{1}{\sqrt{2\pi\hbar}}$$

after integrating by parts and dropping the boundary terms. This is the promised algebraic equation for $\tilde{G}$. Thus, we obtain the formal solution

$$G(x, E) \;=\; \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \mathrm{d}p\; e^{ipx/\hbar}\, \tilde{G}(p, E) \;=\; \frac{m}{\pi\hbar} \int_{-\infty}^{\infty} \mathrm{d}p\; \frac{e^{ipx/\hbar}}{p^2 - \hbar^2 k_E^2}$$

where $k_E \equiv \sqrt{2mE}/\hbar$, as before.

The RHS can be evaluated by contour integration: for $x > 0$, we close the contour in the upper half plane, and for $x < 0$, we close it in the lower half plane. The integrand has poles at $p = \pm\, \hbar k_E$. For $E > 0$, the poles lie on the real axis, and we must choose whether to go over or under them. To reproduce the Green's function for potential scattering discussed above, we go under the $p = \hbar k_E$ pole and over the $p = -\hbar k_E$ pole. Thus, for $x > 0$, the integration is counter-clockwise, and we pick up the $p = \hbar k_E$ pole giving

$$G(x > 0, E) \;=\; i\, \frac{m}{\hbar^2 k_E}\, e^{ik_E x}$$

For $x < 0$, the integration is clockwise, and we pick up the $p = -\sqrt{2mE}$ pole, giving

$$G(x < 0, E) \;=\; i\, \frac{m}{\hbar^2 k_E}\, e^{-ik_E x}$$

Thus,

$$G(x, E) \;=\; i\, \frac{m}{\hbar^2 k_E}\, e^{ik_E |x|}$$

as we found previously. A different prescription for going around the poles will yield a different Green's function, corresponding to different boundary conditions.

For $E < 0$, the poles lie at $p = \pm\, i\hbar\kappa_E$, where $\kappa_E \equiv \sqrt{-2mE}/\hbar$. The most natural choice of contour is along the real axis. Thus, for $x > 0$, we pick up the $p = i\hbar\kappa_E$ pole, giving

$$G(x > 0, E) \;=\; \frac{m}{\hbar^2 \kappa_E}\, e^{-\kappa_E x}$$

Similarly, for $x<0$, we pick up the $p=-i\hbar\kappa_E$ pole, giving

$$G(x<0,E) \;=\; \frac{m}{\hbar^2\,\kappa_E}\,e^{\kappa_E\,x}$$

so that

$$G(x,E) \;=\; \frac{m}{\hbar^2\,\kappa_E}\,e^{-\kappa_E\,|x|}$$

as we found previously.

Problem 4

Consider the wavefunction:

$$\psi(x,t) \;=\; \exp\left[i\,S(x,t)/\hbar\right]$$

If we allow S to be complex, then this ansatz is general so long as ψ is nonvanishing. We apply Schrödinger's equation:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\,\psi(x,t)$$

Thus,

$$-\frac{\partial S}{\partial t}\,e^{i\,S/\hbar} \;=\; -\frac{\hbar^2}{2m}\left(\frac{i}{\hbar}\frac{\partial^2 S}{\partial x^2} - \frac{1}{\hbar^2}\left(\frac{\partial S}{\partial x}\right)^2\right)e^{i\,S/\hbar} + V(x)\,e^{i\,S/\hbar}$$

Cancelling an overall factor of $e^{i\,S/\hbar}$, we find:

$$\frac{i\hbar}{2m}\frac{\partial^2 S}{\partial x^2} \;=\; V(x) + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial S}{\partial t}$$

Suppose that S has a characteristic lengthscale of spatial varations L and a characteristic magnitude S_0. Therefore $\frac{\partial S}{\partial x} \sim S_0/L$ and $\frac{\partial^2 S}{\partial x^2} \sim S_0/L^2$. The $\frac{\partial^2 S}{\partial x^2}$ term is much smaller than the $\left(\frac{\partial S}{\partial x}\right)^2$ term so long as:

$$\frac{\hbar S_0}{2mL^2} \;\ll\; \frac{S_0^2}{2mL^2}$$

or $S_0 \gg \hbar$. Thus, in the large $S/\hbar$ limit, we obtain:

$$0 \;=\; V(x) + \frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + \frac{\partial S}{\partial t}$$

and S may be taken to be real. We recognize this as the classical Hamilton-Jacobi equation for the Hamiltonian $H = \frac{p^2}{2m} + V(x)$, where $p = \frac{\partial S}{\partial x}$.

Now consider a free particle, with $V=0$. Thus,

$$\frac{\partial S}{\partial t} \;=\; -\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2$$

We assume the ansatz:

$$S \;=\; W(x,\alpha) - \frac{\alpha^2}{2m}\,t$$

for some constant α. Thus,

$$\alpha^2 \;=\; \left(\frac{\partial W}{\partial x}\right)^2$$

or

$$W \;=\; \alpha x + S_0$$

where the arbitrary sign is absorbed into the definition of α. Thus,

$$S \;=\; \alpha x - \frac{\alpha^2}{2m} t + S_0$$

and we find the wavefunction:

$$\psi(x,t) \;=\; \exp\left[\frac{i}{\hbar}(p\,x - E\,t) + i\,\phi_0\right]$$

where $p = \alpha$ is the momentum, $E = p^2/2m$ is the energy, and $\phi_0 = S_0/\hbar$ is an arbitrary phase. The solution is exact in this case, since $\frac{\partial^2 S}{\partial x^2}$ vanishes identically.

Problem 5

A coherent state of the 1D SHO is defined by

$$a\,|\lambda\rangle \;=\; \lambda\,|\lambda\rangle$$

where a is the annihilation operator and $\lambda \in \mathbb{C}$.[2]

a)

Consider the state

$$|\lambda\rangle \;=\; e^{-|\lambda|^2/2}\, e^{\lambda a^\dagger}\,|0\rangle$$

We compute

$$a\,|\lambda\rangle \;=\; e^{-|\lambda|^2/2}\, a e^{\lambda a^\dagger}\,|0\rangle \;=\; e^{-|\lambda|^2/2}\,[a, e^{\lambda a^\dagger}]\,|0\rangle \;=\; \lambda e^{-|\lambda|^2/2}\, e^{\lambda a^\dagger}\,|0\rangle \;=\; \lambda\,|\lambda\rangle$$

where we use $a\,|0\rangle = 0$ and $[a, e^{\lambda a^\dagger}] = \lambda\,[a, a^\dagger]\, e^{\lambda a^\dagger}$, as proven in problem 1 of PS #2. Thus, $|\lambda\rangle$ is indeed a coherent state, with a eigenvalue λ. We now check that it is normalized. We have:

$$\langle\lambda|\lambda\rangle \;=\; e^{-|\lambda|^2}\,\langle 0|\, e^{\lambda^\star a}\, e^{\lambda a^\dagger}\,|0\rangle$$

To simplify this expression, we first derive an identity. For operators A and B such that $[A, [A, B]] = [B, [A, B]] = 0$, we already showed that $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$. Applying this identity twice, we find:

$$e^A\, e^B \;=\; e^{A+B}\, e^{\frac{1}{2}[A,B]} \;=\; \left(e^B\, e^A\, e^{-\frac{1}{2}[B,A]}\right) e^{\frac{1}{2}[A,B]} \;=\; e^B\, e^A\, e^{[A,B]}$$

2. Since a is not Hermitean, it can have complex eigenvalues.

(This is the identity suggested in the prompt for part (e).) Applying this identity to $e^{\lambda^\star a} e^{\lambda a^\dagger}$, we find

$$e^{\lambda^\star a} e^{\lambda a^\dagger} = e^{|\lambda|^2} e^{\lambda a^\dagger} e^{\lambda^\star a}$$

Thus,

$$\langle\lambda|\lambda\rangle = \langle 0|\, e^{\lambda a^\dagger} e^{\lambda^\star a}\,|0\rangle = \langle 0|0\rangle = 1$$

so the coherent state is indeed normalized.

b)

We have

$$x = \sqrt{\frac{\hbar}{2m\omega}}\,\big(a + a^\dagger\big) \quad , \quad p = -i\sqrt{\frac{m\hbar\omega}{2}}\,\big(a - a^\dagger\big)$$

Thus, we readily compute (for the state λ):

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,(\lambda + \lambda^\star) \quad , \quad \langle p\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\,(\lambda - \lambda^\star)$$

To compute $\langle x^2\rangle$ and $\langle p^2\rangle$, we rewrite

$$x^2 = \frac{\hbar}{2m\omega}\big(a + a^\dagger\big)^2 = \frac{\hbar}{2m\omega}\big(a^2 + a a^\dagger + a^\dagger a + a^{\dagger 2}\big) = \frac{\hbar}{2m\omega}\big(1 + a^2 + 2a^\dagger a + a^{\dagger 2}\big)$$

and similarly

$$p^2 = -\frac{m\hbar\omega}{2}\big(a - a^\dagger\big)^2 = \frac{m\hbar\omega}{2}\big(-a^2 + a a^\dagger + a^\dagger a - a^{\dagger 2}\big) = \frac{m\hbar\omega}{2}\big(1 - a^2 + 2a^\dagger a - a^{\dagger 2}\big)$$

That is, we place these operators in *normal order* (all a's on the right, all $a^\dagger$'s on the left) which makes the computation of the expectation value easier. We now find:

$$\begin{aligned}\langle x^2\rangle &= \frac{\hbar}{2m\omega}\big(1 + \lambda^2 + 2\,|\lambda|^2 + (\lambda^\star)^2\big) = \frac{\hbar}{2m\omega}\big(1 + (\lambda + \lambda^\star)^2\big)\\ \langle p^2\rangle &= \frac{m\hbar\omega}{2}\big(1 - \lambda^2 + 2\,|\lambda|^2 - (\lambda^\star)^2\big) = \frac{m\hbar\omega}{2}\big(1 - (\lambda - \lambda^\star)^2\big)\end{aligned}$$

Thus,

$$\begin{aligned}\langle(\Delta x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2 = \frac{\hbar}{2m\omega}\big(1 + (\lambda + \lambda^\star)^2 - (\lambda + \lambda^\star)^2\big) = \frac{\hbar}{2m\omega}\\ \langle(\Delta p)^2\rangle &= \langle p^2\rangle - \langle p\rangle^2 = \frac{m\hbar\omega}{2}\big(1 - (\lambda - \lambda^\star)^2 + (\lambda - \lambda^\star)^2\big) = \frac{m\hbar\omega}{2}\end{aligned}$$

and so

$$\langle(\Delta x)^2\rangle\,\langle(\Delta p)^2\rangle = \frac{\hbar^2}{4}$$

which saturates the bound.

c)

We want to express the coherent state in terms of the normalized energy eigenstates

$$|n\rangle \;=\; \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle$$

which is an eigenket of the number operator $N = a^\dagger\, a$ with eigenvalue n. From the definition of $|\lambda\rangle$ we are able to read off

$$|\lambda\rangle \;=\; e^{-|\lambda|^2/2}\sum_{n=0}^{\infty}\frac{\lambda^n\left(a^\dagger\right)^n}{n!}\,|0\rangle \;=\; \sum_{n=0}^{\infty} f(n)\,|n\rangle$$

where

$$f(n) \;=\; \frac{\lambda^n}{\sqrt{n!}}\;e^{-|\lambda|^2/2}$$

The probability of measuring an energy $E_n = \hbar\omega\,(n+1/2)$ is then

$$|f(n)|^2 \;=\; \frac{|\lambda|^n}{n!}\,e^{-|\lambda|^2}$$

which is the Poisson distribution with expectation value $|\lambda|^2$, so that

$$\langle E\rangle \;=\; \hbar\omega\left(|\lambda|^2+1/2\right)$$

However, the expectation value need not be the same as the most probable value (mode)! For instance, the expectation value need not be an eigenvalue, whereas the mode always is. In this case, one can either look up or check by hand that the most probable energy is[3]

$$E_{(\text{mode})} \;=\; \hbar\omega\left(\left\lfloor|\lambda|^2\right\rfloor+1/2\right)$$

where $\lfloor x\rfloor$ denotes the floor, i.e. the greatest integer less than or equal to x.

d)

We wish to compute $e^{-ip\ell/\hbar}\,|0\rangle$ for ℓ real. As before, the best approach is to write the operator in normal ordering, at which point all computations become trivial. We have

$$e^{-ip\ell/\hbar} \;=\; \exp\left[-\ell\sqrt{\frac{m\omega}{2\hbar}}\left(a-a^\dagger\right)\right] \;=\; \exp\left[\lambda\left(a^\dagger-a\right)\right]$$

where

$$\lambda \;=\; \ell\sqrt{\frac{m\omega}{2\hbar}}$$

Applying $e^{A+B} = e^A\,e^B\,e^{-\frac{1}{2}[A,B]}$, we obtain

$$e^{-ip\ell/\hbar} \;=\; e^{\lambda a^\dagger}\,e^{-\lambda a}\,e^{-|\lambda|^2/2}$$

3. For $|\lambda|^2$ an integer, both $E = \hbar\omega\left(|\lambda|^2+1/2\right)$ and $E = \hbar\omega\left(|\lambda|^2-1/2\right)$ have equal probabilities.

Thus

$$e^{-ip\ell/\hbar}\,|0\rangle \;=\; e^{-|\lambda|^2/2}\,e^{\lambda a^\dagger}\,e^{-\lambda a}\,|0\rangle \;=\; e^{-|\lambda|^2/2}\,e^{\lambda a^\dagger}\,|0\rangle \;=\; |\lambda\rangle$$

as desired.

e)

There are several possible approaches to this problem. The simplest is to use the results of part (d), together with the known ground-state wavefunction:

$$\psi_0(x) \;=\; \left(\frac{m\,\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2}\,\frac{m\omega x^2}{\hbar}\right]$$

But then

$$\psi_\lambda(x) \;=\; \langle x|\lambda\rangle \;=\; \langle x|e^{-ip\ell/\hbar}|0\rangle \;=\; \langle x-\ell|0\rangle \;=\; \psi_0(x-\ell)$$

where $\ell = \lambda\sqrt{\frac{2\hbar}{m\omega}}$. Thus,

$$\begin{aligned}\psi_\lambda(x) \;&=\; \left(\frac{m\,\omega}{\pi\hbar}\right)^{1/4}\exp\left[-\frac{1}{2}\,\frac{m\omega}{\hbar}\left(x-\lambda\sqrt{\frac{2\hbar}{m\omega}}\right)^2\right]\\ &=\; \left(\frac{m\,\omega}{\pi\hbar}\right)^{1/4} e^{-\lambda^2}\exp\left[-\frac{1}{2}\,\frac{m\omega x^2}{\hbar}+\sqrt{\frac{2m\omega}{\hbar}}\,\lambda x\right]\end{aligned}$$

However, this procedure is only valid for real λ, since we implicitly assumed that ℓ was real.

To establish the analogous result for complex λ, we follow the procedure suggested in the prompt. The defining equation $a\,|\lambda\rangle = \lambda\,|\lambda\rangle$ becomes a differential equation when written in terms of the wavefunction. We have

$$a \;=\; \frac{m\omega x + ip}{\sqrt{2m\hbar\omega}}$$

Thus,

$$m\omega x\,\psi_\lambda(x) + \hbar\,\frac{\partial}{\partial x}\,\psi_\lambda(x) \;=\; \sqrt{2m\hbar\omega}\,\lambda\,\psi_\lambda(x)$$

or

$$\left(x-\sqrt{2}\,x_0\,\lambda\right)\psi_\lambda(x) + x_0^2\,\frac{\partial}{\partial x}\,\psi_\lambda(x) \;=\; 0$$

where $x_0 \equiv \sqrt{\hbar/m\omega}$. The solution is

$$\psi_\lambda(x) \;=\; \mathcal{N}\exp\left[-\frac{1}{2x_0^2}\left(x-\sqrt{2}\,x_0\,\lambda\right)^2\right]$$

for $\lambda \in \mathbb{C}$. We compute the normalization:

$$\int_{-\infty}^{\infty}|\psi_\lambda(x)|^2\,\mathrm{d}x \;=\; |\mathcal{N}|^2\int_{-\infty}^{\infty}\exp\left[-\frac{1}{x_0^2}\left(x-\sqrt{2}\,x_0\,\mathrm{Re}\,\lambda\right)^2 + 2\,(\mathrm{Im}\,\lambda)^2\right]\mathrm{d}x \;=\; |\mathcal{N}|^2\,x_0\,\sqrt{\pi}\,e^{2\,(\mathrm{Im}\,\lambda)^2}$$

Thus,

$$\mathcal{N} \;=\; \frac{1}{x_0^{1/2}\,\pi^{1/4}}\,e^{-(\operatorname{Im}\lambda)^2+i\phi}$$

where ϕ is an unknown phase factor, so that

$$\begin{aligned}
\psi_\lambda(x) \;&=\; \frac{1}{x_0^{1/2}\,\pi^{1/4}}\,e^{-(\operatorname{Im}\lambda)^2+i\phi}\exp\!\left[-\frac{1}{2x_0^2}\left(x-\sqrt{2}\,x_0\,\lambda\right)^2\right]\\
&=\; \frac{1}{x_0^{1/2}\,\pi^{1/4}}\,e^{-(\operatorname{Im}\lambda)^2-\lambda^2+i\phi}\exp\!\left[-\frac{x^2}{2x_0^2}+\sqrt{2}\,\lambda\,\frac{x}{x_0}\right]\\
&=\; \frac{1}{x_0^{1/2}\,\pi^{1/4}}\;e^{-\lambda^2/2}\,e^{-|\lambda|^2/2}\,e^{i\,\tilde{\phi}}\exp\!\left[-\frac{x^2}{2x_0^2}+\sqrt{2}\,\lambda\,\frac{x}{x_0}\right]
\end{aligned}$$

where we absorb an extra phase into ϕ on the last line. Note that

$$\langle\lambda'\,|\lambda\rangle \;=\; e^{-|\lambda'|^2/2-|\lambda|^2/2}\,\langle 0|\,e^{\lambda'^\star a}\,e^{\lambda a^\dagger}\,|0\rangle \;=\; e^{-|\lambda'|^2/2-|\lambda|^2/2}\,e^{\lambda'^\star\lambda}\,\langle 0|\,e^{\lambda a^\dagger}e^{\lambda'^\star a}\,|0\rangle \;=\; e^{-|\lambda'|^2/2-|\lambda|^2/2}\,e^{\lambda'^\star\lambda}$$

where we apply $e^A\,e^B = e^B\,e^A\,e^{[A,B]}$ as in part (a). Thus, we find

$$\begin{aligned}
\int_{-\infty}^{\infty}\psi^\star_{\lambda'}(x)\,\psi_\lambda(x)\,\mathrm{d}x \;&=\; \frac{1}{x_0\sqrt{\pi}}\;e^{-\lambda^2/2-\lambda'^{\star 2}/2}\,e^{-|\lambda'|^2/2-|\lambda|^2/2}\,e^{i\,(\tilde{\phi}_\lambda-\tilde{\phi}_{\lambda'})}\\
&\quad\times\int_{-\infty}^{\infty}\exp\!\left[-\frac{x^2}{x_0^2}+\sqrt{2}\,(\lambda'^\star+\lambda)\,\frac{x}{x_0}\right]\\
&=\; [\ldots]\times\int_{-\infty}^{\infty}\exp\!\left[-\left(\frac{x}{x_0}-\frac{1}{\sqrt{2}}\,(\lambda'^\star+\lambda)\right)^2+\frac{1}{2}\,(\lambda'^\star+\lambda)^2\right]\\
&=\; e^{-\lambda^2/2-\lambda'^{\star 2}/2}\,e^{-|\lambda'|^2/2-|\lambda|^2/2}\,e^{i\,(\tilde{\phi}_\lambda-\tilde{\phi}_{\lambda'})}\,e^{(\lambda'^\star+\lambda)^2/2}\\
&=\; e^{-|\lambda'|^2/2-|\lambda|^2/2}\,e^{i\,(\tilde{\phi}_\lambda-\tilde{\phi}_{\lambda'})}\,e^{\lambda'^\star\lambda}
\end{aligned}$$

Thus, we must have $\tilde{\phi}_\lambda = \tilde{\phi}_{\lambda'}$, so we may fix $\tilde{\phi}_\lambda = 0$ (since it is independent of λ). In net, we find

$$\psi_\lambda(x) \;=\; \frac{1}{x_0^{1/2}\,\pi^{1/4}}\;e^{-\lambda^2/2}\,e^{-|\lambda|^2/2}\exp\!\left[-\frac{x^2}{2x_0^2}+\sqrt{2}\,\lambda\,\frac{x}{x_0}\right]$$

which matches that given in the problem assignment after accounting for the different normalization conventions.

Now we want to examine how the coherent state evolves with time. Since

$$|\lambda\rangle \;=\; \sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\;e^{-|\lambda|^2/2}\,|n\rangle$$

we see that

$$e^{-iHt/\hbar}\,|\lambda\rangle \;=\; \sum_{n=0}^{\infty}\frac{\lambda^n}{\sqrt{n!}}\;e^{-i\omega n t}\,e^{-|\lambda|^2/2}\,|n\rangle \;=\; \sum_{n=0}^{\infty}\frac{\left(e^{-i\omega t}\,\lambda\right)^n}{\sqrt{n!}}\;e^{-|\lambda|^2/2}\,|n\rangle \;=\; \left|\lambda e^{-i\omega t}\right\rangle$$

Thus,

$$\psi_\lambda(x,t) \;=\; \psi_{\lambda(t)}(x)$$

where $\lambda(t) = \lambda e^{-i\omega t}$.

We rewrite the time-dependent wavefunction as

$$\psi_\lambda(x,t) \;=\; \frac{1}{x_0^{1/2}\,\pi^{1/4}}\,\exp\left[-\frac{1}{2x_0^2}\Big(x-\sqrt{2}\,x_0\,\mathrm{Re}\,\lambda(t)\Big)^2+\sqrt{2}\,i\,(\mathrm{Im}\,\lambda(t))\,\frac{x}{x_0}\right]$$

In this form, it is clear that $\psi_\lambda(x,t)$ is a wave packet of fixed width, which oscillates back and forth with central position $x_{\mathrm{cen}}(t) = \sqrt{2}\,x_0\,\mathrm{Re}\,\lambda(t)$. The x-dependent phase (the second term in the exponential) represents a time varying momentum $p_{\mathrm{cen}}(t) = \sqrt{2}\,\frac{\hbar}{x_0}\,\mathrm{Im}\,\lambda(t)$. In particular, taking $\lambda(t) = \lambda_0\,e^{-i\omega(t-t_0)}$ with λ_0 real, we find

$$\begin{aligned} x_{\mathrm{cen}} &= \sqrt{\frac{2\hbar}{m\omega}}\,\lambda_0\cos\left[\omega\,(t-t_0)\right] \\ p_{\mathrm{cen}} &= -\sqrt{2m\hbar\omega}\,\lambda_0\sin\left[\omega\,(t-t_0)\right] \end{aligned}$$

so the position and momentum of the wave packet oscillate 90° out of phase with $p = m\,\dot{x}$, as one would expect of a classical particle.

Problem 6

The Hamiltonian for a charged particle in the absence of electric or magnetic fields is:

$$H \;=\; \frac{|\boldsymbol{p}|^2}{2m}+V(\boldsymbol{x})$$

where $V(\boldsymbol{x})$ is some unspecified potential.

Suppose we incorporate a vector potential via the prescription $\boldsymbol{p}\to\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}$. Thus,

$$H \;=\; \frac{1}{2m}\left(|\boldsymbol{p}|^2-\frac{e}{c}\,(\boldsymbol{p}\cdot\boldsymbol{A}+\boldsymbol{A}\cdot\boldsymbol{p})+\frac{e^2}{c^2}\,|\boldsymbol{A}|^2\right)$$

Since in general $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{A}$ does not commute with $\boldsymbol{p}$. Suppose that the magnetic field $\boldsymbol{B}$ is constant. One can check that the vector potential

$$\boldsymbol{A} \;=\; \frac{1}{2}\,\boldsymbol{B}\times\boldsymbol{x}$$

obeys $\nabla\times\boldsymbol{A} = \boldsymbol{B}$. Thus, the Hamiltonian becomes:

$$\begin{aligned} H &= \frac{1}{2m}\left(|\boldsymbol{p}|^2-\frac{e}{2c}\,\left(\boldsymbol{p}\cdot(\boldsymbol{B}\times\boldsymbol{x})+(\boldsymbol{B}\times\boldsymbol{x})\cdot\boldsymbol{p}\right)+\frac{e^2}{4c^2}\,|\boldsymbol{B}\times\boldsymbol{x}|^2\right)+V(\boldsymbol{x}) \\ &= \frac{|\boldsymbol{p}|^2}{2m}-\frac{e}{4mc}\left[\boldsymbol{B}\cdot(\boldsymbol{x}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{x})\right]+\frac{e^2}{8mc^2}\left[|\boldsymbol{B}|^2\,|\boldsymbol{x}|^2-(\boldsymbol{B}\cdot\boldsymbol{x})^2\right]+V(\boldsymbol{x}) \\ &= \frac{|\boldsymbol{p}|^2}{2m}-\frac{e}{2mc}\,(\boldsymbol{B}\cdot\boldsymbol{L})+\frac{e^2}{8mc^2}\left[|\boldsymbol{B}|^2\,|\boldsymbol{x}|^2-(\boldsymbol{B}\cdot\boldsymbol{x})^2\right]+V(\boldsymbol{x}) \end{aligned}$$

where in the last line we use $\boldsymbol{x}\times\boldsymbol{p} = -\,\boldsymbol{p}\times\boldsymbol{x}$, since

$$\begin{aligned} \varepsilon_{ijk}\,(x_j\,p_k+p_j\,x_k) &= \varepsilon_{ijk}\,[x_j, p_k] \\ &= 0 \end{aligned}$$

The second term in the Hamiltonian is simply $-\boldsymbol{\mu}\cdot\boldsymbol{B}$, where

$$\boldsymbol{\mu} \;=\; \frac{e}{2mc}\boldsymbol{L}$$

is the magnetic moment induced by the orbital angular momentum.

Now consider the third term, and write $\boldsymbol{B}=B\,\hat{\boldsymbol{z}}$ WLOG. We find:

$$\ldots \;=\; \frac{e^2B^2}{8mc^2}\left[x^2+y^2\right]$$

At first, the appearance of this term may seem slightly suprising, as we began with a setup which was (potentially) invariant under translations $\boldsymbol{x}\rightarrow\boldsymbol{x}+\Delta\boldsymbol{x}$ (assuming that $V(\boldsymbol{x})$ is a constant), whereas this term is manifestly not invariant under translations (it looks like a 2D simple harmonic oscillator potential).

However, upon writing $\boldsymbol{A}$ in the form

$$\boldsymbol{A} \;=\; \frac{1}{2}\boldsymbol{B}\times\boldsymbol{x}$$

we have made a gauge choice which is not invariant under general translations $\boldsymbol{x}\rightarrow\boldsymbol{x}+\Delta\boldsymbol{x}$, but only under z translations (and rotations in the x-y plane). The canonical momentum $\boldsymbol{p}_{\text{can}}=\boldsymbol{p}_{\text{phys}}+\frac{e}{c}\boldsymbol{A}$ is not gauge invariant, hence the appearance of other terms in the Hamiltonian which are not gauge invariant. However, in cases where $V(\boldsymbol{x})$ breaks x and y translation symmetry itself, this gauge choice is rather natural.

Problem 7

Start with the Lagrangian for a free particle:

$$L \;=\; \frac{1}{2}m\,|\dot{\boldsymbol{x}}|^2$$

We add an interaction term (for a particle of charge $+e$):

$$L \;=\; \frac{1}{2}m\,|\dot{\boldsymbol{x}}|^2+\frac{e}{c}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A}(\boldsymbol{x})$$

The Euler-Lagrange equations give:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}_i} \;=\; \frac{\partial L}{\partial x_i}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[m\,\dot{x}_i+\frac{e}{c}A_i(x(t))\right] \;&=\; \frac{e}{c}\dot{x}_j\,\partial_i A_j\\ &=\; m\,\ddot{x}_i+\frac{e}{c}\dot{x}_j\,\partial_j A_i\end{aligned}$$

and so

$$m\,\ddot{x}_i \;=\; \frac{e}{c}\dot{x}_j\left(\partial_i A_j-\partial_j A_i\right) \;=\; \frac{e}{c}\dot{x}_j\,\varepsilon_{ijk}\,\varepsilon_{klm}\,\partial_l A_m \;=\; \frac{e}{c}\varepsilon_{ijk}\,\dot{x}_j\,B_k$$

or

$$m \frac{\mathrm{d}^2 \boldsymbol{x}}{\mathrm{d}t^2} \;=\; \frac{e}{c}\, \dot{\boldsymbol{x}} \times \boldsymbol{B}$$

which is the Lorentz force law for magnetostatics.

Starting with this Lagrangian, one can obtain the Hamiltonian of the previous problem by the standard procedure. The canonical momentum is:

$$\boldsymbol{p}_{\mathrm{can}} \;=\; \frac{\partial L}{\partial \dot{\boldsymbol{x}}} \;=\; m\,\dot{\boldsymbol{x}} + \frac{e}{c}\,\boldsymbol{A} \;=\; \boldsymbol{p}_{\mathrm{phys}} + \frac{e}{c}\,\boldsymbol{A}$$

as we saw before. Thus,

$$\begin{aligned} H \;=\; \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L \;&=\; \Big(m\,\dot{\boldsymbol{x}} - \frac{e}{c}\,\boldsymbol{A}\Big)\cdot\dot{\boldsymbol{x}} - \frac{1}{2}\,m\,|\dot{\boldsymbol{x}}\,|^2 + \frac{e}{c}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A}(\boldsymbol{x}) \\ &=\; \frac{1}{2}\,m\,|\dot{\boldsymbol{x}}\,|^2 \;=\; \frac{1}{2m}\,|\boldsymbol{p}_{\mathrm{phys}}|^2 \;=\; \frac{1}{2m}\,|\boldsymbol{p} - \frac{e}{c}\,\boldsymbol{A}|^2 \end{aligned}$$

In the above solution, we assumed that $\boldsymbol{A}$, and hence $\boldsymbol{B}$, was independent of time. If we allow $\boldsymbol{B}$ to be time-dependent, then $\boldsymbol{E}$ will be nonvanishing in general, with $\boldsymbol{E} = -\,\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi$. The $\frac{\partial \boldsymbol{A}}{\partial t}$ term which we omitted above will contribute the addition $e\boldsymbol{E}$ term to the Lorentz force law, as expected (we should also include $-\,e\,\phi(x)$ in the Lagrangian in this case.)