P6572 HW #2
Due September 9, 2011

1. If A and B are two operators that do not commute with each other but which both commute with $[A, B]$, they satisfy
$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}.$$
   (a) To prove this first show that $[B, e^{xA}] = e^{xA}[B, A]x$. Next define $G(x) = e^{xA}e^{xB}$, and show that
   $$\frac{dG}{dx} = (A + B + [A, B]x)G.$$
   Integrate this to obtain the desired result.

   (b) More generally, show that for arbitrary A and B
   $$\lim_{\alpha,\beta\to 0} e^{\alpha A}e^{\beta B} = e^{\alpha A+\beta B+\frac{1}{2}\alpha\beta[A,B]+X}$$
   where X is of higher order in α, β.

2. Sakurai, p.66, problem 29

   (a) Verify that for all functions $F(\mathbf{x})$ and $G(\mathbf{p})$ that can be expressed as power series in their arguments that
   $$[x_i, G(\mathbf{p})] = i\hbar\frac{\partial G}{\partial p_i}, \qquad [p_i, F(\mathbf{x})] = -i\hbar\frac{\partial F}{\partial x_i}$$
   (b) Evaluate $[x^2, p^2]$.

   (c) The classical Poisson bracket is defined as
   $$[A, B] = \sum_j \frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j}$$
   Compare your result from part b) with the classical Poisson bracket $[x^2, p^2]_{classical}$.

3. Sakurai, p.67 problem 30
The translation operator for a finite (spatial) displacement is given by
$$\mathcal{T}(\mathbf{l}) = \exp\left(\frac{-i\mathbf{p}\cdot\mathbf{l}}{\hbar}\right),$$
where $\mathbf{p}$ is the momentum *operator*.

   (a) Evaluate
   $$[x_i, \mathcal{T}(\mathbf{l})].$$
   (b) Using (a) (or otherwise), demonstrate how the expectation value of $\langle\mathbf{x}\rangle$ changes under translation.

4. **Canonical transformation**

We know that for any function $F_1(q,\bar{q})$, we can generate a canonical transformation and that $K = H + \frac{\partial F_1}{\partial t}$. But suppose that we are handed a transformation $(\bar{p}(q,\bar{q}), p(q,\bar{q}))$, how can we determine if it is canonical? We could try to find the generating function $F(q,\bar{q})$ such that $\frac{\partial F}{\partial q} = p$, and $\frac{\partial F}{\partial \bar{q}} = -\bar{p}$. Alternatively, supposing for simplicity that $\frac{\partial F}{\partial t} = 0$ so that $H = K$, and generalizing $q, \bar{q}, p, \bar{p}$ to vectors, we have that

$$\dot{\bar{q}}_i = \frac{\partial \bar{q}_i}{\partial q_j}\dot{q}_j + \frac{\partial \bar{q}_i}{\partial p_j}\dot{p}_j = \frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial \bar{q}_i}{\partial p_j}\frac{\partial H}{\partial q_j} \equiv [\bar{q}_i, H] \tag{1}$$

and

$$\frac{\partial H}{\partial p_j} = \frac{\partial K}{\partial p_j} = \frac{\partial H}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial p_j} + \frac{\partial H}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial p_j} \tag{2}$$

$$\frac{\partial H}{\partial q_j} = \frac{\partial K}{\partial q_j} = \frac{\partial H}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial q_j} + \frac{\partial H}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial q_j} \tag{3}$$

Next substitute Equations 2 and 3 into Equation 1

$$\begin{aligned}
\dot{\bar{q}}_i &= \frac{\partial \bar{q}_i}{\partial q_j}\left(\frac{\partial H}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial p_j} + \frac{\partial H}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial p_j}\right) - \frac{\partial \bar{q}_i}{\partial p_j}\left(\frac{\partial H}{\partial \bar{q}_k}\frac{\partial \bar{q}_k}{\partial q_j} + \frac{\partial H}{\partial \bar{p}_k}\frac{\partial \bar{p}_k}{\partial q_j}\right) \\
&= \frac{\partial H}{\partial \bar{q}_k}\left(\frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial \bar{q}_k}{\partial p_j} - \frac{\partial \bar{q}_i}{\partial p_j}\frac{\partial \bar{q}_k}{\partial q_j}\right) + \frac{\partial H}{\partial \bar{p}_k}\left(\frac{\partial \bar{q}_i}{\partial q_j}\frac{\partial \bar{p}_k}{\partial p_j} - \frac{\partial \bar{q}_i}{\partial p_j}\frac{\partial \bar{p}_k}{\partial q_j}\right) \\
&= \frac{\partial H}{\partial \bar{q}_k}[\bar{q}_i, \bar{q}_k] + \frac{\partial H}{\partial \bar{p}_k}[\bar{q}_i, \bar{p}_k]
\end{aligned}$$

We see that we recover Hamilton's equations if $[\bar{q}_i, \bar{q}_k] = 0$ and $[\bar{q}_i, \bar{p}_k] = \delta_{ik}$. The transformation $\bar{q}(q,p), \bar{p}(q,p)$ is canonical if $[\bar{q}_i, \bar{q}_k] = 0$ and $[\bar{q}_i, \bar{p}_k] = \delta_{ik}$, where the Poisson bracket of A and B is

$$[A, B] = \sum_j \frac{\partial A}{\partial q_j}\frac{\partial B}{\partial p_j} - \frac{\partial A}{\partial p_j}\frac{\partial B}{\partial q_j}$$

All canonical transformations satisfy the fundamental Poisson bracket relationship.

Generalizing to $2n$ dimensional phase space, for two arbitrary functions, the Poisson bracket with respect to q, p is independent of the canonical variables that it is expressed in.

$$[F, G] = \sum_{j=1}^{n}\left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_i}\right)$$

In particular,

$$\frac{\partial \bar{q}}{\partial q}\frac{\partial \bar{p}}{\partial p} - \frac{\partial \bar{q}}{\partial p}\frac{\partial \bar{q}}{\partial p} = 1$$

is invariant.

Show directly that the transformation

$$Q = \log\left(\frac{1}{q}\sin p\right), \qquad P = q\cot p$$

is canonical.

5. **Momentum operator in position space** (Sakurai, p.67, problem 33)

   (a) Prove the following:

   i.
   $$\langle p' \mid x \mid \alpha \rangle = i\hbar \frac{\partial}{\partial p'} \langle p' \mid \alpha \rangle,$$

   ii.
   $$\langle \beta \mid x \mid \alpha \rangle = \int dp' \; \phi_\beta^*(p') i\hbar \frac{\partial}{\partial p'} \phi_\alpha(p'),$$

   where $\phi_\alpha(p') = \langle p' \mid \alpha \rangle$ and $\phi_\beta(p') = \langle p' \mid \beta \rangle$ are momentum-space wave functions.

   (b) What is the physical significance of

   $$\exp\left(\frac{ix\Xi}{\hbar}\right),$$

   where x is the position operator and Ξ is some number with the dimension of momentum? Justify your answer.

6. **Harmonic Oscillator Action**
   Consider the harmonic oscillator, for which the general solution is

   $$x(t) = A\cos\omega t + B\sin\omega t$$

   Express the energy in terms of A and B and note that it does not depend on time. Now choose A and B such that $x(0) = x_1$ and $x(T) = x_2$. Write down the energy in terms of x_1, x_2, and T. Show that the action for the trajectory connecting x_1 and x_2 is

   $$S_{cl}(x_1, x_2, T) = \frac{m\omega}{2\sin\omega T}[(x_1^2 + x_2^2)\cos\omega T - 2x_1x_2].$$

   Verify that

   $$\frac{\partial S_{cl}}{\partial t} = -E.$$

   where

   $$S_{cl} = \int_0^T L dt$$

7. **Hermitian matrices**
   Assume that any hermitian matrix can be diagnoalized by a unitary matrix. From this, show that the necessary and sufficient condition that two hermitian matrices can be diagonalized by the same unitary transformation is that they commute.