P6572 HW #11
Due December 5, 2011

1. **Sakurai, p. 350, problem 18**
Work out the *quadratic* Zeeman effect for the ground-state hydrogen atom $[\langle \mathbf{x} \mid 0\rangle = (1/\sqrt{\pi a_0^3})e^{-r/a_0}]$ due to the usually neglected $e^2\mathbf{A}^2/2m_e c^2$-term in the Hamiltonian taken to first order. Write the energy shift as
$$\Delta = -\frac{1}{2}\chi \mathbf{B}^2$$
and obtain an expression for *diamagnetic susceptibility*, χ. (The following definite integral may be useful:
$$\int_0^\infty e^{-\alpha r} r^n dr = \frac{n!}{\alpha^{n+1}}.)$$

2. **Sakurai, p. 348, problem 12**
This is similar to the problem we did in class. A system that has three unperturbed states can be represented by the perturbed Hamiltonian matrix
$$\begin{pmatrix} E_1 & 0 & a \\ 0 & E_1 & b \\ a^* & b^* & E_2 \end{pmatrix}$$
where $E_2 > E_1$. The quantities a and b are to be regarded as perturbations that are of the same order and are small compared with $E_2 - E_1$. Use the second-order nondegenerate perturbation theory to calculate the perturbed eigenvalues. (Is this procedure correct?) Then diagonalize the matrix to find the exact eigenvalues. Finally, use the second-order degenerate perturbation theory. Compare the three results obtained.

3. **Sakurai, p. 349, problem 13**
Compute the Stark effect for the $2S_{1/2}$ and $2P_{1/2}$ levels of hydrogen for a field ϵ sufficiently weak so that $e\epsilon a_0$ is small compared to the fine structure, but take the Lamb shift δ (δ=1057 MHz) into account (that is, ignore $2P_{1/2}$ in this calculation). Show that for $e\epsilon a_z \ll \delta$, the energy shifts are quadratic in ϵ, whereas for $e\epsilon a_0 \gg \delta$ they are linear in ϵ. (The radial integral you need is $\langle 2s \mid r \mid 2p\rangle = 3\sqrt{3}a_0$.) Briefly discuss the consequences (if any) of time reversal for this problem.

4. **Sakurai, p. 353, problem 28**
A hydrogen atom in its ground state $[(n, l, m = (1, 0, 0)]$ is placed between the plates

of a capacitor. A time-dependent but spatially uniform electric field (not potential!) is applied as follows:

$$\mathbf{E} = \begin{cases} 0 & \text{for } t < 0. \\ \mathbf{E}_0 e^{-t/\tau} & \text{for } t > 0 \end{cases}, \quad (\mathbf{E}_0 \text{ in the positive z} - \text{direction})$$

Using first order time dependent perturbation theory, compute the probability for the atom to be found at $t \gg \tau$ in each of the three $2p$ states: $(n, l, m) = (2, 1, \pm 1 \text{ or } 0)$. Repeat the problem for the $2s$ state: $(n, l, m) = (2, 0, 0)$. You need not attempt to evaluate radial integrals, but perform all other integrations (with respect to angles and time).

5. **Sakurai, p. 354, problem 33**
Consider the spontaneous emission of a photon by an excited state. The process is known to be an E1 transition. Suppose the magnetic quantum number of the atom decreases by one unit. What is the angular distribution of the emitted photon? Also discuss the polarization of the photon with attention to angular-momentum conservation for the whole (atom plus photon) system.

6. **Sakurai, p. 355, problem 36**
Derive an expression for the density of free particle states in *two* dimensions, that is, the two-dimensional analog of

$$\rho(E)dEd\Omega = \left(\frac{L}{2\pi}\right)^3 \left(\frac{mk}{\hbar^2}\right) dEd\Omega, \left(\mathbf{k} \equiv \frac{\mathbf{p}}{\hbar}, E = \frac{\mathbf{p}^2}{2m}\right).$$

Your answer should be written as a function of k (or E) times $dEd\phi$, where ϕ is the polar angle that characterizes the momentum direction in two dimensions.

7. **Sakurai, p. 356, problem 38**
Linearly polarized light of angular frequency ω is incident on a one electron "atom" whose wave function can be approximated by the ground state of a three-dimensional isotropic harmonic oscillator of angular frequency ω_0. Show that the differential cross section for the ejection of a photoelectron is given by

$$\frac{d\sigma}{d\Omega} = \frac{4\alpha\hbar^2 k_f^3}{m^2\omega\omega_0}\sqrt{\frac{\pi\hbar}{m\omega_0}} \exp\left\{-\frac{\hbar}{m\omega_0}\left[k_f^2 + \left(\frac{\omega}{c}\right)^2\right]\right\} \times \sin^2\theta\cos^2\phi \exp\left[\left(\frac{2\hbar k_f \omega}{m\omega_0 c}\right)\cos\theta\right]$$

provided the ejected electron of momentum $\hbar k_f$ can be regarded as being in a plane-wave state. (The coordinate system used is shown in Figure 5.10, p. 341)

8. **Sakurai, p. 356, problem 40**
Obtain an expression for $\tau(2p \to 1s)$ for the hydrogen atom. Verify that it is equal to 1.6×10^{-9} s.

9. **Hyperfine Structure**

The hyperfine structure of a level is caused by the interactions of the magnetic moment of the nucleus with both the orbital and the spin magnetic moments of the electrons. We shall assume in the following that the hyperfine splitting is much smaller than the fine structure splitting. As a result of this interaction, the angular momentum J of the electrons adds on to that of the nucleus I to give a resultant angular momentum F, which is the only true constant of the motion. However J^2 is still almost a constant of the motion. Each fine structure level is then split into a hyperfine multiplet with values of f ranging from $|j - i|$ to $j + i$.

One can show (See Cohen-Tanoudjii, Chapter XII, and Griffiths chapter 6), that for a hydrogen atom in the ground state, the hyperfine interaction is

$$\frac{16\pi}{3\hbar^2}\mu_B\mu_N g_p(\mathbf{S}\cdot\mathbf{I})\delta^3(\mathbf{r})$$

where I is the spin of the nucleus and g_p is the anomalous magnetic moment of the proton. S is the spin of the electron. Because we are in the ground state, $S = J$ because $L = 0$.

Consider in particular the hyperfine structure of the ground state of hydrogen. Compute the frequency (in megahertz) and the wavelength (in cm) of the radiation emitted in a transition between the two hyperfine levels. This is the famous hydrogen line of radioastronomy. Compute the lifetime of the upper level, replacing the electric dipole moment operator of the usual formula with the magnetic moment operator (this is an $M1$ transition). How does the calculated lifetime fit with your knowledge that this line is actually observed?