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1 Path Integral

Suppose we have the propagator

$$K(x_f, t_f, x_0, t_0) = \langle x_f, t_f \mid x_0, t_0 \rangle = \left\langle x_f \mid e^{-iH(t_f - t_0)/\hbar} \mid x_0 \right\rangle$$

We can just as easily take two steps

$$K(x_f, t_f, x_0, t_0) = \langle x_f, t_f \mid x_0, t_0 \rangle = \left\langle x_f \mid e^{-iH(t_f - t_1)/\hbar} e^{-iH(t_1 - t_0)/\hbar} \mid x_0 \right\rangle$$

or we could divide the total time T into N steps, with $\delta = T/N$. Then

$$K(x_f, t_f, x_0, t_0) = \langle x_f, t_f \mid x_0, t_0 \rangle = \left\langle x_f \mid e^{-iH\delta/\hbar} e^{-iH\delta/\hbar} ... \mid x_0 \right\rangle$$

and then we could insert the identity everywhere along the path.

$$\begin{aligned} K(x_f, t_f, x_0, t_0) &= \langle x_f \mid e^{-iH\delta/\hbar} \int dx_{N-1} \mid x_{N-1} \rangle \langle x_{N-1} \mid e^{-iH\delta/\hbar} \int dx_{N-2} \mid x_{N-2} \rangle \langle x_{N-2} \mid \dots \\ &\dots \int dx_1 \mid x_1 \rangle \langle x_1 \mid e^{-iH\delta/\hbar} \mid x_0 \rangle \\ &= \int dx_{N-1} K(x_f, t_f, x_{N-1}, t_{N-1}) \int dx_{N-2} K(x_{N-1}, t_{N-1}, x_{N-2}, t_{N-2}) \dots \\ &\dots \int dx_1 K(x_1, t_1, x_0, t_0) \end{aligned}$$

The amplitude is the sum of all N-legged paths.

$$A = \sum_{\text{paths}} A_{\text{paths}}$$

$$\sum_{\text{paths}} = \int dx_1 dx_2 ... dx_{N-1}, \qquad A_{\text{path}} = K_{x_N, x_{N-1}} K_{x_{N-1}, x_{N-2}} ...$$

Let's consider the j^{th} term.

$$K(x_{j+1}, x_j) = \left\langle x_{j+1} \mid e^{-iH\delta/\hbar} \mid x_j \right\rangle$$

Since δ is small we can expand the exponential and we have

$$K(x_{j+1}, x_j) \sim \left\langle x_{j+1} \mid 1 - iH\delta/\hbar + O(\delta^2) \mid x_j \right\rangle$$

Then we can insert the identity $\int dp_j |\, p_j\rangle\langle p_j \mid$ and

$$\begin{aligned}
K(x_{j+1}, x_j) &\sim \int dp_j \left\langle x_{j+1} \mid 1 - iH\frac{\delta}{\hbar} \mid p_j \right\rangle \langle p_j \mid x_j\rangle \\
&\sim \int dp_j \langle x_{j+1} \mid p_j\rangle\langle p_j \mid x_j\rangle - i \left\langle x_{j+1} \mid H\frac{\delta}{\hbar} \mid p_j \right\rangle \langle p_j \mid x_j\rangle \\
&\sim \int dp_j \langle x_{j+1} \mid p_j\rangle\langle p_j \mid x_j\rangle - i\frac{\delta}{\hbar}(\frac{p_j^2}{2m} + V(x_{j+1}))\langle x_{j+1} \mid p_j\rangle\langle p_j \mid x_j\rangle \\
&\sim \int \frac{dp_j}{2\pi\hbar} \left(e^{i(x_{j+1}-x_j)p_j/\hbar} - i\frac{\delta}{\hbar}(\frac{p_j^2}{2m} + V(x_{j+1}))e^{i(x_{j+1}-x_j)p_j/\hbar} \right) \\
&\sim \int \frac{dp_j}{2\pi\hbar} e^{i(x_{j+1}-x_j)p_j/\hbar} \exp\left(-i\frac{\delta}{\hbar}(\frac{p_j^2}{2m} + V(x_{j+1})) \right) \\
&\sim \int \frac{dp_j}{2\pi\hbar} e^{i(x_{j+1}-x_j)p_j/\hbar} \exp\left(-i\frac{\delta}{\hbar}H \right)
\end{aligned}$$

Now we write $(x_{j+1} - x_j)/\delta = \dot{x}_j$ and we have

$$K(x_{j+1}, x_j) \sim \int \frac{dp_j}{2\pi\hbar} e^{i\delta\dot{x}_j p_j/\hbar} e^{-i\frac{\delta}{\hbar}H}$$

There are N such factors in the amplitude so

$$A_{\text{path}} = \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} \exp\left(i\frac{\delta}{\hbar} \sum_{j=0}^{N-1} (\dot{x}_j p_j - H(x_j, p_j)) \right)$$

That's the amplitude for one path. Now integrate over all paths

$$K(x_N, x_0) = \int \prod_{j=1}^{N-1} dx_j \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} \exp i\frac{\delta}{\hbar} \sum_{j=0}^{N-1} (\dot{x}_j p_j - H(x_j, p_j))$$

As $N \to \infty$ the sum becomes an integral over all time and we write

$$K(x_N, x_0) = \int \mathcal{D}x(t) \int \mathcal{D}p(t) \exp i\frac{\delta}{\hbar} \int_0^T dt(\dot{x}p - H(x,p))$$

This is the phase space path integral. If the Hamiltonian has the standard form $H = \frac{p^2}{2m} + V(x)$ then we can integrate each of the terms in the sum

$$K(x_N, x_0) = \int \prod_{j=1}^{N-1} dx_j \exp(-i\frac{\delta}{\hbar} \sum_{j=0}^{N-1} V(x_j)) \int \prod_{j=0}^{N-1} \frac{dp_j}{2\pi\hbar} \exp i\frac{\delta}{\hbar} \sum_{j=0}^{N-1} (\dot{x}_j p_j - \frac{p_j^2}{2m})$$

We use

$$\int_{-\infty}^{\infty} e^{-\alpha x^2 - \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha}$$

where the above holds for pure imaginary α if it is regarded as a limit, namely if $\alpha = a + ib, a > 0$ it is the limit as $a \to 0$. This is what it looks like

$$\int \frac{dp}{2\pi\hbar} \exp i\frac{\delta}{\hbar}(\dot{x}p - \frac{p^2}{2m}) = \sqrt{\frac{m}{2\pi i\frac{\delta}{\hbar}}} e^{i\frac{\delta}{\hbar}m\dot{x}^2/2}$$

Putting it all together

$$\begin{aligned} K &= \int \prod_{j=1}^{N-1} dx_j \exp\left(-i\frac{\delta}{\hbar} \sum_{j=0}^{N-1} V(x_j)\right) \prod_{j=0}^{N-1} \left(\sqrt{\frac{m}{2\pi\hbar i\delta}} \exp i\frac{\delta}{\hbar}\frac{m\dot{x}_j^2}{2}\right) \\ &= \left(\frac{m}{2\pi\hbar i\delta}\right)^{N/2} \int \prod_{j=1}^{N-1} dx_j \exp i\frac{\delta}{\hbar} \sum_{j=0}^{N-1} \left(\frac{m\dot{x}_j^2}{2} - V(x_j)\right) \end{aligned}$$

The sum is an approximation of the action of a path passing through the points $x_0, x_1, x_2, ...$

$$K = \int \mathcal{D}x(t) e^{iS[x(t)]}$$

is the configuration space path integral.

1.1 Free particle path integral

The configuration space path integral for a free particle is

$$K = \left(\frac{m}{2\pi\hbar i\delta}\right)^{N/2} \int \prod_{j=1}^{N-1} dx_j \exp\left[\frac{i\delta}{\hbar} \sum_{j=0}^{N-1} \left(\frac{m\dot{x}_j^2}{2}\right)\right]$$

$$K = \Big(\frac{m}{2\pi\hbar i\delta}\Big)^{N/2} \int \prod_{j=1}^{N-1} dx_j \exp\left[\frac{i\delta}{\hbar}\sum_{j=0}^{N-1}\frac{m}{2}\left(\frac{(x_{j+1}-x_j)}{\delta}\right)^2\right]$$

$$K = \Big(\frac{m}{2\pi\hbar i\delta}\Big)^{N/2} \int \prod_{j=1}^{N-1} dx_j \exp\left[\frac{im}{2\delta\hbar}\sum_{j=0}^{N-1}\left((x_{j+1}-x_j)\right)^2\right]$$

$$K = \Big(\frac{m}{2\pi\hbar i\delta}\Big)^{N/2} \int \prod_{j=1}^{N-1} dx_j \exp\left[\frac{im}{2\delta\hbar}\big((x_N-x_{N-1})^2+(x_{N-1}-x_{N-2})^2+...+(x_1-x_0)^2\big)\right]$$

where x_0 and x_N are initial and final points. The integrals are Gaussian and can be evaluated exactly but since they are coupled it ain't pretty. Let's see if we can figure it out. First let's define $y_i = (\frac{m}{2\delta\hbar})^{\frac{1}{2}} x_i$. Then

$$K = \Big(\frac{m}{2\pi\hbar i\delta}\Big)^{N/2} \int \prod_{j=1}^{N-1} \left(\frac{2\delta\hbar}{m}\right)^{\frac{1}{2}} dy_j \exp\left[i\left((y_N-y_{N-1})^2+(y_{N-1}-y_{N-2})^2+...+(y_1-y_0)^2\right)\right]$$

$$K = \Big(\frac{m}{2\pi\hbar i\delta}\Big)^{N/2} \left(\frac{2\delta\hbar}{m}\right)^{(N-1)/2} \int \prod_{j=1}^{N-1} dy_j \exp\left[i\left((y_N-y_{N-1})^2+(y_{N-1}-y_{N-2})^2+...+(y_1-y_0)^2\right)\right]$$

Let's do the y_1 integration first.

$$\begin{aligned}
\int dy_1 \exp i\left((y_2-y_1)^2+(y_1-y_0)^2\right) &= \int dy_1 \exp i(y_2^2+y_0^2+2y_1^2-2y_1(y_2+y_0)) \\
&= \int dy_1 \exp i(\sqrt{2}y_1-(y_2+y_0)/\sqrt{2})^2 \\
&\quad \times \exp i(y_2^2+y_0^2)\exp -\frac{i}{2}(y_2+y_0)^2 \\
&= \int \frac{dz}{\sqrt{2}}\exp -\frac{1}{i}(z)^2 \exp \frac{i}{2}(y_2-y_0)^2 \\
&= \sqrt{\frac{i\pi}{2}}\exp \frac{i}{2}(y_2-y_0)^2
\end{aligned}$$

Next we do the y_2 integration.

$$\begin{aligned}
&\sqrt{\frac{i\pi}{2}}\int dy_2 \exp -\frac{1}{i}\left((y_3-y_2)^2+\frac{1}{2}(y_2-y_0)^2\right) \\
= \ &\sqrt{\frac{i\pi}{2}}\int dy_2 \exp -\frac{1}{2i}\left(3y_2^2-2y_2(2y_3+y_0)\right)\exp -\frac{1}{i}(y_3^2+\frac{1}{2}y_0)^2
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{i\pi}{2}} \int \frac{dz}{\sqrt{3}} \exp -\frac{1}{2i}\left(z^2 - (2y_3 + y_0)^2/3\right) \exp -\frac{1}{i}(y_3^2 + \frac{1}{2}y_0)^2 \\
&= \sqrt{\frac{i\pi}{2}}\sqrt{\frac{2i\pi}{3}} \exp \frac{1}{2i}\left((2y_3 + y_0)^2/3\right) \exp -\frac{1}{i}(y_3^2 + \frac{1}{2}y_0)^2 \\
&= \sqrt{\frac{(i\pi)^2}{3}} \exp -\frac{1}{3i}(y_3 - y_0)^2
\end{aligned}
$$

It looks like it goes to

$$\sqrt{\frac{(i\pi)^{N-1}}{N}} \exp\left(-\frac{1}{Ni}(y_N - y_0)^2\right)$$

Putting it all together we have

$$K = \lim_{N\to\infty} \left(\frac{m}{2\pi\hbar i\delta}\right)^{N/2} \left(\frac{2\delta\hbar}{m}\right)^{(N-1)/2} \frac{(i\pi)^{\frac{N-1}{2}}}{N^{\frac{1}{2}}} \exp\left(-\frac{1}{Ni}(y_n - y_0)^2\right)$$

The answer is

$$K = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar\delta}\right)^{N/2} \frac{1}{\sqrt{N}} \left(\frac{2\pi i\hbar\delta}{m}\right)^{(N-1)/2} e^{im(x'-x)^2/2N\hbar\delta}$$

$$K = \lim_{N\to\infty} \left(\frac{m}{2\pi i\hbar N\delta}\right)^{1/2} e^{im(x'-x)^2/2N\hbar\delta}$$

Since $N\delta = T$ we have

$$K = \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} e^{im(x'-x)^2/2\hbar T}$$

which is of course the same as we calculated directly. Now on further investigation we see that

$$K = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}} \exp i\frac{m}{2}\left(\frac{x_N - x_0}{T}\right)^2 T = \left(\frac{m}{2\pi i\hbar T}\right)^{\frac{1}{2}} \exp(\frac{i}{\hbar}\int_0^T L_{cl}dt)$$

Cute huh. The coordinate space path integral for the free particle, the sum of the action through every possible point in space, reduces to simply the classical action. The propagator reduces to two factors, one being the phase $\exp \frac{i}{\hbar}S_{cl}$

1.2 Harmonic oscillator path integral

The coordinate space path integral for the harmonic oscillator is

$$K = \Big(\frac{m}{2\pi\hbar i\delta}\Big)^{N/2} \int \prod_{j=}^{N-1} dx_j \exp i\delta \sum_{j=0}^{N-1} \left(\frac{m\dot{x}_j^2}{2} - \frac{1}{2}m\omega^2 x_j^2\right)$$

Now let's write

$$x(t) = x_{cl}(t) + y(t), \quad dx = dy, \qquad \dot{x} = \dot{x}_{cl} + \dot{y}$$

Then

$$\begin{aligned} & \sum \left(\frac{m\dot{x}^2}{2} - \frac{1}{2}m\omega^2 x_j^2\right) \\ = \quad & \left(\frac{m\dot{x}_{cl}^2}{2} - \frac{1}{2}m\omega^2 x_{cl}^2\right) + \sum \left(\frac{\partial L}{\partial \dot{x}} \mid_{\dot{x}_{cl}} \dot{y} + \frac{\partial L}{\partial x} \mid_{x_{cl}} y\right) + \sum \left(\frac{m\dot{y}^2}{2} - \frac{1}{2}m\omega^2 y^2\right) \end{aligned}$$

Let's look at the middle term and convert the sum to an integral

$$\int dt \left(\frac{\partial L}{\partial x}y + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}y\right) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}y\right) = \frac{\partial L}{\partial \dot{x}}y|_{t_0}^{t_N} + \int dt \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\right) y = 0$$

The first term is zero because $y(_0) = y(t_N) = 0$. So

$$K = \lim_{N\to\infty} \Big(\frac{m}{2\pi\hbar i\delta}\Big)^{N/2} \exp\frac{i}{\hbar}S_{cl} \int \prod_{j=1}^{N-1} dy_j \exp i\delta \sum_{j=0}^{N-1} \left(\frac{m\dot{y}^2}{2} - \frac{1}{2}m\omega^2 y_j^2\right)$$

The PI over y is independent of the endpoints. It is zero at each end. It will depend only on the total time T

$$K = \exp\frac{i}{\hbar}S_{cl}Y(T), \qquad Y(T) = \Big(\frac{m\omega}{2\pi i \sin\omega T}\Big)^{1/2}$$

and if

$$x(t) = A\cos(\omega t) + B\sin\omega t, x_N = A\cos\omega T + B\sin\omega T, \qquad x_0 = A$$

Then $B = (x_N - x_0\cos\omega T)/\sin\omega T$

$$S_{cl} \;=\; \int \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}mx^2\right) dt$$

$$
\begin{aligned}
&= \frac{1}{2}\int dt\,\left(m(-\omega A\sin\omega t+\omega B\cos\omega t)^2-m\omega^2(A\cos\omega t+B\sin\omega t)^2\right)\\
&= \frac{1}{2}\int dt\; m\omega^2(A^2\sin^2\omega t+B^2\cos^2\omega t-2AB\sin\omega t\cos\omega t\\
&\quad -(A^2\cos^2\omega t+B^2\sin^2\omega t+2AB\sin\omega t\cos\omega t))\\
&= \frac{1}{2}\int dt\; m\omega^2((B^2-A^2)\cos 2\omega t-2AB\sin 2\omega t)\\
&= \frac{m\omega}{4}((B^2-A^2)\sin 2\omega t+2AB\cos 2\omega t)\;|_0^T\\
&= \frac{m\omega}{4}((B^2-A^2)\sin 2\omega T+2AB(\cos 2\omega T-1))\\
&= \frac{m\omega}{4}((x_N-x_0\cos\omega T)^2-x_0^2\sin^2\omega T)\frac{\sin 2\omega T}{\sin^2\omega T}\\
&\quad +2x_0(x_n-x_0\cos\omega T)(\frac{\cos 2\omega T-1}{\sin\omega T}))\\
&= \frac{m\omega}{4}((x_N^2+x_0^2\cos 2\omega T-2x_Nx_0\cos\omega T)\frac{2\sin\omega T\cos\omega T}{\sin^2\omega T}\\
&\quad +2x_0(x_n-x_0\cos\omega T)(\frac{\cos 2\omega T-1}{\sin\omega T}))\\
&= \frac{m\omega}{4}((x_N^2+x_0^2\cos 2\omega T-2x_Nx_0\cos\omega T)\frac{2\cos\omega T}{\sin\omega T}\\
&\quad +2x_0(x_n-x_0\cos\omega T)(\frac{\cos 2\omega T-1}{\sin\omega T}))\\
&= \frac{m\omega}{4}((2x_N^2\cos\omega T+2x_0^2\cos\omega T-4x_Nx_0)\frac{1}{\sin\omega T}\\
&= \frac{m\omega}{2\sin\omega T}((x_N^2+x_0^2)\cos\omega T-2x_Nx_0)
\end{aligned}
$$

1.3 Principle of Least Action

Consider the configuration space path integral

$$K=\int \mathcal{D}x(t)e^{iS[x(t)]/\hbar}.$$

It says that a particle going from initial to final position and time takes all possible paths. The classical path is included but it gets no special mention. Every path has precisely unit magnitude. The contributions from the classical path and the totally wild path are the same. It turns out that the amplitudes interfere with each

other in a very special way. Consider two neighboring paths $x(t)$ and $x'(t)$ and let $x'(t) = x(t) + \eta(t)$, with $\eta(t)$ small. Then we can write the action

$$S[x'] = S[x + \eta] = S[x] + \int dt\eta(t)\frac{\delta S[x]}{\delta x(t)} + O(\eta^2)$$

The contribution of the two paths to the PI is

$$A \sim e^{iS[x]/\hbar}\left(1 + \exp\frac{i}{\hbar}\int dt\eta(t)\frac{\delta S[x]}{\delta x(t)}\right)$$

The phase difference between the two paths is $\frac{1}{\hbar}\int dt\eta(t)\frac{\delta S[x]}{\delta x(t)}$. Smaller $\hbar$ larger phase difference. Even paths that are very close together will have large phase difference for small $\hbar$ and on average they will interfere destructively.

This is true except for one exceptional path, that which extremizes the action, namely the classical path $x_c(t)$. For this path

$$S[x_c + \eta] = S[x_c] + \mathcal{O}(\eta^2).$$

The classical path and a close neighbor will have actions which differ by much less than two randomly chosen but equally close paths.

If the problem is classical (action $\gg \hbar$), paths near the classical path will on average interfere constructively (small phase difference) whereas for random paths the interference will be on average destructive. Classically, the particles motion is governed by the principle that the action is stationary.