Physics 443, Solutions to PS 8

1.**Griffiths 5.7**. For notational simplicity, we define $\psi_1 \doteq \psi_a, \psi_2 \doteq \psi_b, \psi_3 \doteq \psi_c$. Then we can write:

$$\text{Distinguishable } \psi(\{x_i\}) = \prod_{i=1}^{3} \psi_i(x_i)$$

$$\text{Bosons } \psi(\{x_i\}) = \sum_{i,j,k=1}^{3} \psi_a(x_i)\psi_b(x_j)\psi_c(x_k)(\epsilon_{ijk})^2$$

$$\text{Fermions } \psi(\{x_i\}) = \sum_{i,j,k=1}^{3} \psi_a(x_i)\psi_b(x_j)\psi_c(x_k)(\epsilon_{ijk})$$

2.**Griffiths 5.11**. This question is solving a messy integral. Lets tackle the physics i.e. part(b) first. Since

$$V_{ee} = \frac{e^2}{4\pi\epsilon_0}\langle\frac{1}{|r_1 - r_2|}\rangle,$$

using part (a) we can estimate that $V_{ee} = -5\epsilon_1/2 = +34\text{ev}$. Therefore an approximation to the energy of ground state of Helium would be $-109+34 = -75\text{ev}$, which compares nicely to the experimental value of -79ev.

Now to the integral of part (a). Using the choice of coordinates recommended by Griffiths, we need to solve the following integral.

$$I = \int d^3r_1 \, d^3r_2 \, \left(\frac{8}{\pi a^3}\right)^2 \frac{\exp(-4(r_1+r_2)/a)}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2}}.$$

We proceed as follows. First we scale our variables r_1 and r_2 to remove the length-scale a. Then we perform the integral over θ_2. Notice that as a function of θ_2, we simply have a square-root form of the integrand (since $\sin\theta_2 \, d\theta_2 = -d(\sin\theta_2)$). Also the $d\phi_2$ integral is trivial giving us a 2π. So we can rewrite our integral in the form

$$2\pi\left(\frac{64}{\pi^2 a}\right)\int d^3r_1 \, r_2^2 \, dr_2 \, \exp(-4(r_1+r_2))\left(\frac{r_1+r_2}{r_1r_2} - \frac{|r_1-r_2|}{r_1r_2}\right).$$

Notice that this last bit is different depending on which of r_1 and r_2 is greater. In particular, if $0 < r_2 < r_1$ it gives $2/r_1$, and for $r_1 < r_2 < \infty$ we have $2/r_2$. This means that we can rearrange our integral as follows:

$$4\pi\left(\frac{64}{\pi^2 a}\right)\int d^3r_1 \, \exp(-4r_1)\left(\int_0^{r_1} dr_2 \, \frac{r_2^2}{r_1}e^{-4r_2} + \int_{r_1}^{\infty} dr_2 \, r_2 e^{-4r_2}\right).$$

The dr_2 integrals can now be done. We turn to the d^3r_1 integral, where the angular part just gives us 4π. We are left with just the radial integral as follows:

$$I = \left(\frac{32}{a}\right)\int_0^\infty dr_1 \left(r_1 e^{-4r_1} - 2r_1^2 e^{-8r_1} - r_1 e^{-8r_1}\right) = \frac{5}{4a}$$

3.**Griffiths 5.18**.

(a) Using

$$\psi(x) = A\sin(kx) + B\cos(kx), \quad (0 < x < a) \tag{1}$$
$$\text{and}$$
$$A\sin(ka) = [e^{iKa} - \cos(ka)]B, \tag{2}$$

show that the wave function for a particle in the periodic delta function potnetial can be written in the form

$$\psi(x) = C[\sin(kx) + e^{-iKa}\sin k(a-x)], \quad (0 \le x \le a).$$

[We can solve Equation 2 for A and substitute into Equation 1 to get

$$\begin{aligned}
\psi(x) &= \frac{B}{\sin(ka)}\left((e^{iKa} - \cos(ka))\sin(kx) + \cos(kx)\sin(ka)\right) \\
&= \frac{Be^{iKa}}{\sin(ka)}\left(\sin(kx) + e^{-iKa}(\cos(kx)\sin(ka) - \cos(ka)\sin(kx))\right) \\
&= \frac{Be^{iKa}}{\sin(ka)}\left(\sin(kx) + e^{-iKa}\sin k(a-x)\right) \\
&= C\left(\sin(kx) + e^{-iKa}\sin k(a-x)\right)
\end{aligned}$$

(b) We see from Equation 2 that if $ka = z = n\pi$ then either $B = 0$, or $e^{iKa} = \cos(n\pi) = \pm 1$. Then substitution into the equation matching the derivative of the wavefunction.

$$kA - e^{-iKa}k[A\cos(ka) - B\sin(ka)] = -2m\alpha B/\hbar^2 \tag{3}$$

becomes
$$kA - e^{-iKa}kA\cos(n\pi) = -2m\alpha B/\hbar^2 \tag{4}$$

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and therefore $B = 0$. So the wave function at the top of the band is

$$\psi(x) = A \sin(kx)$$

and it is zero at each delta function.

4.**Griffiths 5.20**. If there are delta function *wells* instead of spikes then $V(x) = \sum_i^{N-1} \alpha \delta(x - ia)$, $\alpha < 0$, and we have to consider solutions with $E < 0$. Then the general solution is

$$\psi(x) = A \sinh(\kappa x) + B \cosh(\kappa x) \quad (0 < x < a) \tag{5}$$

where $\kappa = \sqrt{-2mE}/\hbar$ and the wave function to the left of the origin is

$$\psi(x - a) = e^{-iKa}[A \sinh(\kappa x) + B \cosh(\kappa x)] \tag{6}$$

Continuity at $x = 0$ implies

$$e^{-iKa}[A \sinh(\kappa a) + B \cosh(\kappa a)] = B \tag{7}$$

The boundary condition for the derivatives gives

$$\kappa A - \kappa e^{-iKa}[A \cosh(\kappa a) + B \sinh(\kappa a)] = \frac{2m\alpha}{\hbar^2} B \tag{8}$$

We solve Equation 7 for A and substitute into Equation 8 and get

$$\frac{2m\alpha}{\kappa\hbar^2} B \sinh(\kappa a) = B(e^{iKa} - \cosh(\kappa a)) - e^{-iKa}\left(B(e^{iKa} - \cosh(\kappa a))\cosh(\kappa a) + B\sinh^2(\kappa a)\right)$$

$$\frac{2m\alpha}{\kappa\hbar^2} \sinh(\kappa a) = e^{iKa} - 2\cosh(\kappa a) - e^{-iKa}\left((-\cosh(\kappa a))\cosh(\kappa a) + \sinh^2(\kappa a)\right)$$

$$= 2\cos Ka - 2\cosh(\kappa a)$$

$$\rightarrow \quad \cos Ka = \cosh(\kappa a) + \frac{m\alpha}{\kappa\hbar^2}\sinh(\kappa a)$$

$$\rightarrow \quad f(z) = \cosh(z) + \beta \frac{\sinh(z)}{z}$$

where $z = -\kappa a$ and $\beta = m\alpha a/\hbar^2$. $f(z)$ for the negative energy solutions with delta function wells, ($\beta = -1.5$) is plotted in Figure 1. $f(z)$ for the positive energy states with delta function wells is shown in Figure 2. We see that the first band is has the same number of states as he subsequent bands. Some of these states are at energies less than zero, and the remainder of the band is filled by states with energies greater than zero.
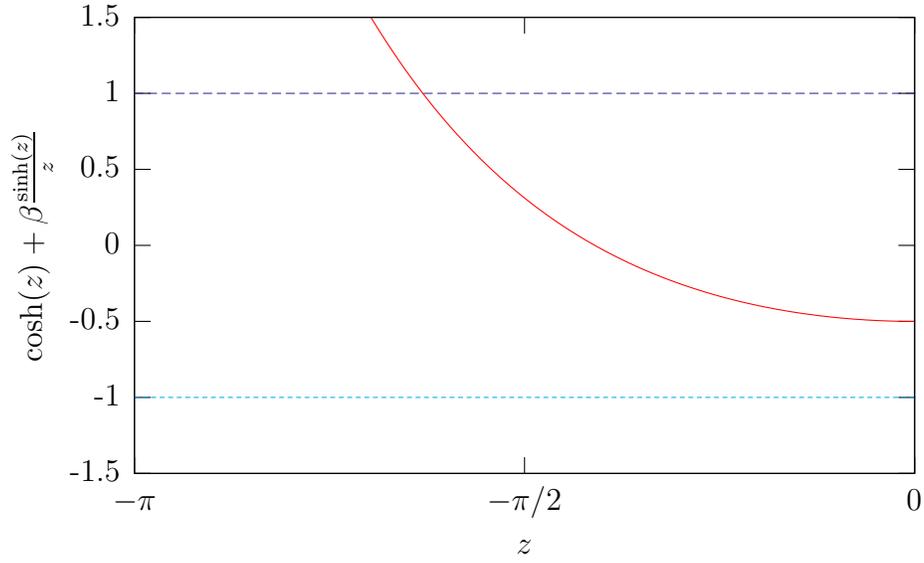
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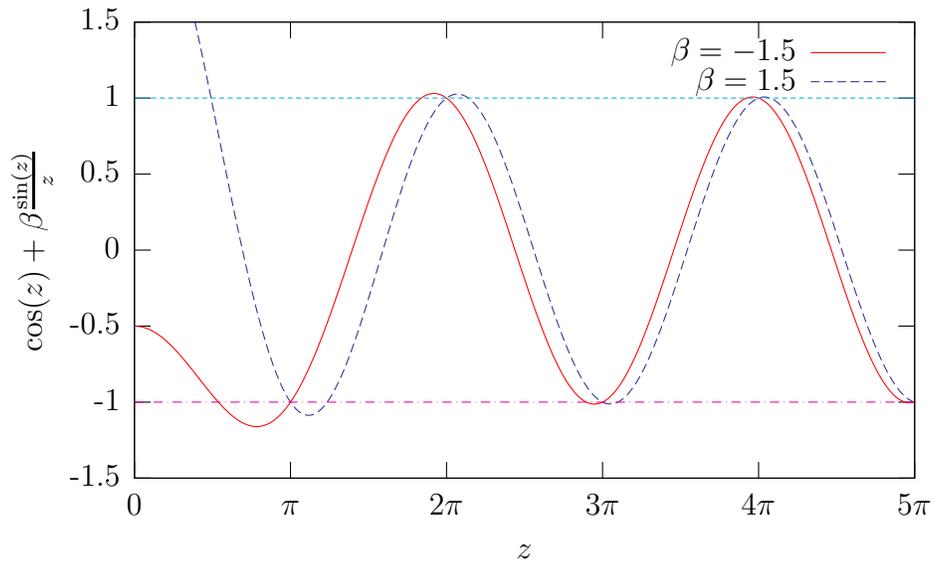Figure 1: Negative energy states, $\beta = -1.5$

Figure 2: Positive energy states, $\beta = \pm 1.5$. The red line is $f(z)$ for delta function wells, and the blue dashed line for delta function spikes.

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5.**Griffiths 5.23**.

(a) Because the energy $E = 9/2\ \hbar\omega$, and $(n_x + n_y + n_z + 3/2)\hbar\omega = E$, we see that $n_x + n_y + n_z = 3$. For distinguishable particles, we have the following particle configurations: $(0, 3, 0, 0, 0 \cdots), d = 1$, and $(2, 0, 0, 1, 0 \cdots), d = 3$, and $(1, 1, 1, 0, 0 \cdots), d = 6$. Here d is the degeneracy of that configuration. Check that for each configuration, the sum of the entries is the number of particles $N = 3$. Also check that the sum of the degeneracy $\sum d = 1 + 3 + 6 = 10$ is what we would expect from $d_3 = (n + 1)(n + 2)/2 = (4)(5)/2 = 10$. The probability of getting $E = \hbar\omega/2$ is the sum of the contribution from the second configuration $(3/10)(2/3)$ plus the contribution from the third configuration $(1/3)(6/10)$, giving $P(\hbar\omega/2) = (2/5)$. Similarly you can verify that $P(3\hbar\omega/2) = 3/10$, $P(5\hbar\omega/2) = 1/5$, $P(7\hbar\omega/2) = 1/10$.

(b) For the case of Fermions, there is only one allowed configuration $(1, 1, 1, 0 \cdots)$. We have that with $P = 1/3$, each of the states $2E = \hbar\omega, 3\hbar\omega$, and $5\hbar\omega$ are possible.

(c) For Bosons, the three configurations of part (a) are equally likely, giving us $P(\hbar\omega/2) = 1/3, P(3\hbar\omega/2) = 4/9, P(5\hbar\omega/2) = 1/9, P(7\hbar\omega/2) = 1/9$.

6.**Griffiths 5.28**. Evaluate the integrals

$$N = \frac{V}{2\pi^2} \int_0^\infty \frac{k^2}{e^{[(\hbar^2 k^2/2m)-\mu]/k_B T} \pm 1} dk \tag{9}$$

$$E = \frac{V}{2\pi^2} \frac{\hbar^2}{2m} \int_0^\infty \frac{k^4}{e^{[(\hbar^2 k^2/2m)-\mu]/k_B T} \pm 1} dk \tag{10}$$

for the case of identical fermions at absolute zero. [At $T = 0$, the chemical potential $\mu = E_F$. Then at $T = 0$, Equations 9 and 10 become

$$\rho = \frac{1}{2\pi^2} \int_0^{k_F} k^2 dk = \frac{1}{6\pi^2} k_F^3 \tag{11}$$

$$E = \frac{V}{2\pi^2} \frac{\hbar^2}{2m} \int_0^{k_F} k^4 dk = \frac{V}{10\pi^2} \frac{\hbar^2}{2m} k_F^5 \tag{12}$$

With the substitution $k_F = \left(\frac{2mE_F}{\hbar^2}\right)^{\frac{1}{2}}$, Equations 11 and 12 become

$$\rho = \frac{1}{6\pi^2} \left(\frac{2mE_F}{\hbar^2}\right)^{\frac{3}{2}} \Rightarrow E_F = (6\pi^2\rho)^{\frac{2}{3}} \frac{\hbar^2}{2m}$$

$$E = \frac{V}{10\pi^2}\frac{\hbar^2}{2m}\left(\frac{2mE_F}{\hbar^2}\right)^{\frac{5}{2}}$$

$$= \frac{V}{10\pi^2}\frac{\hbar^2}{2m}\left(6\pi^2\rho\right)^{\frac{5}{3}}$$

$$= \frac{1}{10\pi^2}\frac{\hbar^2}{2m}\left(6\pi^2 Nq\right)^{\frac{5}{3}} V^{-2/3}$$

which agree with our analysis of free electrons in a solid where we found that

$$E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \tag{13}$$

$$E_{tot} = \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m}V^{-2/3}] \tag{14}$$

(Note that for electrons there is an extra factor of 2 in Equations 13 and 14, to account for the spin degeneracy.)

7.Griffiths 5.29

(a) Show that for bosons the chemical potential must always be less than the minimum allowed energy. $Hint:$ $n(\epsilon)$ cannot be negative.

[For bosons, the most probable number of particles in a given state with energy ϵ is

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}$$

We see that $n(\epsilon) < 0$ if $\mu > \epsilon$.]

(b) In particular, for the ideal bose gas, $\mu(T) < 0$ for all T. Show that in this case $\mu(T)$ monotonically increases as T decreases, assuming N and V are held constant. $Hint$: Study Equation 9, with the minus sign.

[Consider the exponent $(\hbar^2 k^2/2m - \mu)/k_B T$. If $T \to 0$, and $\mu < 0$, then the exponent is infinite for all k, and $N/V \to 0$. The ratio N/V can only be preserved if $\mu(T) \to 0$ as $T \to 0$.]

(c) A crisis (called **Bose condensation**) occurs when (as we lower T) $\mu(T)$ hits zero. Evaluate the integral, for $\mu = 0$, and obtain the formula for the critical temperature T_c at which this happens. Below the

critical temperature, the particles crowd into the ground state, and the calculational device of replacing the discrete sum

$$\sum_{n=1}^{\infty} N_n = N,$$

by a continuous integral (Equation 9) loses its validity. $Hint$:

$$\int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s)\zeta(s),$$

where Γ is Euler's **gamma function** and ζ is the **Riemann zeta function**. Look up the appropriate numerical values.

[In the limit where $\mu = 0$, Equation 9 for bosons is

$$\rho = \frac{1}{2\pi^2} \int_0^{\infty} \frac{k^2}{e^{[(\hbar^2 k^2/2m)]/k_B T} - 1} dk$$

Let $x = (\hbar^2 k^2/2mk_B T)$ and the integral becomes

$$\rho = \frac{1}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \int_0^{\infty} \frac{x^{\frac{3}{2}-1}}{e^x - 1} dx$$

$$= \frac{1}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \Gamma(\frac{3}{2})\zeta(\frac{3}{2})$$

$$= \frac{1}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2}\right)^{3/2} \frac{\sqrt{\pi}}{2}(2.61238)]$$

(d) Find the critical temperatiure for ^4He. Its density, at this temperature, is 0.15gm/cm^3. $Comment$: The experimental value of the critical temperature in ^4He is 2.17 K.

[Solve for

$$T = \left(\frac{4\pi^2 \rho}{(\sqrt{\pi}/2)(2.61238)}\right)^{2/3} \frac{\hbar^2}{2mk_B}$$

$$= \left(\frac{4\pi^2(150 \text{ kg/ m}^3)/(6.64 \times 10^{-27}\text{kg})}{(\sqrt{\pi}/2)(2.61238)}\right)^{2/3}$$

$$\times \frac{(1.054 \times 10^{-34})^2(\text{ J} - \text{ s})^2}{2(6.64 \times 10^{-27} \text{ kg})(1.38 \times 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1})}$$

$$= 3.2 \text{ K}]$$