P443 HW #8
Due March 26, 2008

Identical Particles

1. **Griffiths 5.7**. Suppose you had *three* particles, one in state $\psi_a(x)$, one in state $\psi_b(x)$, and one in state $\psi_c(x)$. Assuming ψ_a, ψ_b, and ψ_c are orthonormal, construct the three-particle states, (analogous to

$$\begin{aligned} \psi(x_1, x_2) &= \psi_a(x_1)\psi_b(x_2) \\ \psi_\pm(x_1, x_2) &= \frac{1}{\sqrt{2}}[\psi_a(x_1)\psi_b(x_2) \pm \psi_a(x_2)\psi_b(x_1)]) \end{aligned}$$

representing (a) distinguishable particles, (b) identical bosons, and (c) identical fermions. Keep in mind that (b) must be completely symmetric, under interchange of *any* pair of particles, and (c) must be completely *anti*symmetric, in the same sense. *Comment* : A handy way to construct completely antisymmetric wave functions is to form the Slater determinant, whose first row is $\psi_a(x_1), \psi_b(x_1), \psi_c(x_1)$, etc., whose second row is $\psi_a(x_2), \psi_b(x_2), \psi_c(x_2)$, etc. and so on (this device works for any number of particles).

2. **Griffiths 5.11**

   (a) Calculate $\langle(1/|\mathbf{r}_1 - \mathbf{r}_2|)\rangle$ for the state

   $$\psi_0 = \psi_{100}(\mathbf{r}_1)\psi_{100}(\mathbf{r}_2) = \frac{8}{\pi a^3}e^{-2(r_1+r_2)/a}.$$

   Hint : Do the $d^3\mathbf{r}_2$ integral first, using spherical coordinates, and setting the polar axis along $\mathbf{r}_1$, so that

   $$|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta_2}.$$

   The θ_2 integral is easy, but be careful to take the *positive root*. You'll have to break the r_2 integral into two pieces, one ranging from 0 to r_1, the other from r_1 to ∞. *Answer* : $5/4a$.

   (b) Use your result in (a) to estimate the electron interaction energy in the ground state of helium. Express your answer in electron volts and add it to $E_0 = 8(-13.6 \text{ eV}) = -109$ eV to get a corrected estimate of the ground state energy. Compare the experimental value.

3. **Griffiths 5.18**

   (a) Using

   $$\begin{aligned} \psi(x) &= A\sin(kx) + B\cos(kx), \quad (0 < x < a) \\ &\text{and} \\ A\sin(ka) &= [e^{iKa} - \cos(ka)]B, \end{aligned}$$

   show that the wave function for a particle in the periodic delta function potnetial can be written in the form

   $$\psi(x) = C[\sin(kx) + e^{-iKa}\sin k(a-x)], \quad (0 \le x \le a).$$

   (Don't bother to determine the normalization constant C.)

   (b) There is an exception: At the top of a band, where z is an integer multiple of π, (a) yields $\psi(x) = 0$. Find the correct wave function for this case. Note what happens to ψ at each delta function.

4. Delta function spikes, Griffiths 5.20

5. **Griffiths 5.23**. Suppose you had three (noninteractng) particles, in thermal equilibrium, in a one-dimensional harmonic oscillator potential, with a total energy $E = (9/2)\hbar\omega$.

   (a) If they are distinguishable particles (but all with the same mass), what are the possible occupation-number configurations, and how many distinct (three-particle) states are there for each one? What is the most probable configuration? If you picked a particle at random and measured its energy, what values might you get, and what is the probability of each one? What is the most probable energy?

   (b) Do the same for the case of identical fermions (ignoring spin).

   (c) Do the same for the case of identical bosons (ignoring spin).

6. **Griffiths 5.28**. Evaluate the integrals

$$N = \frac{V}{2\pi^2}\int_0^\infty \frac{k^2}{e^{[(\hbar^2k^2/2m)-\mu]/k_BT} \pm 1}dk \tag{1}$$

$$E = \frac{V}{2\pi^2}\frac{\hbar^2}{2m}\int_0^\infty \frac{k^4}{e^{[(\hbar^2k^2/2m)-\mu]/k_BT} \pm 1}dk \tag{2}$$

for the case of identical fermions at absolute zero. Compare your results with our analysis of free electrons in a solid where we found that

$$\begin{aligned} E_F &= \frac{\hbar^2}{2m}(3\rho\pi^2)^{2/3} \\ E_{tot} &= \frac{\hbar^2(3\pi^2 Nq)^{5/3}}{10\pi^2 m} V^{-2/3} \end{aligned}$$

(Note that for electrons there is an extra factor of 2 in Equations 1 and 2, to account for the spin degeneracy.)

7. **Griffiths 5.29**

   (a) Show that for bosons the chemical potential must always be less than the minimum allowed energy. *Hint* : $n(\epsilon)$ cannot be negative.

   (b) In particular, for the ideal bose gas, $\mu(T) < 0$ for all T. Show that in this case $\mu(T)$ monotonically icnreases as T decreases, assuming N and V are held constant. *Hint* : Study Equation 1, with the minus sign.

   (c) A crisis (called **Bose condensation**) occurs when (as we lower T) $\mu(T)$ hits zero. Evaluate the integral, for $\mu = 0$, and obtain the formula for the critical temperature T_c at which this happens. Below the critical temperature, the particles crowd into the ground state, and the calculational device of replacing the discrete sum

   $$\sum_{n=1}^{\infty} N_n = N,$$

   by a continuous integral (Equation 1) loses its validity. *Hint* :

   $$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Gamma(s)\zeta(s),$$

   where Γ is Euler's **gamma function** and ζ is the **Riemann zeta function**. Look up the appropriate numerical values.

   (d) Find the critical temperatiure for ${}^4He$. Its density, at this temperature, is 0.15gm/cm^3. *Comment* : The experimental value of the critical temperature in ${}^4He$ is 2.17 K.