Physics 443 HW #7
Due March 12, 2008

1. **Griffiths 4.50**. Suppose two spin-1/2 particles are known to be in the singlet configuration.

$$(|\,00\rangle = \frac{1}{\sqrt{2}}(|+\frac{1}{2}\rangle|-\frac{1}{2}\rangle - |-\frac{1}{2}\rangle|+\frac{1}{2}\rangle)$$

   Let $S_a^{(1)}$ be the component of the spin angular momentum of particle number 1 in the direction defined by the unit vector $\hat{a}$. Similarly, let $S_b^{(2)}$ be the component of the 2's angular momentum in the direction $\hat{b}$. Show that

$$\langle S_a^{(1)} S_b^{(2)}\rangle = -\frac{\hbar^2}{4}\cos\theta,$$

   where θ is the angle between $\hat{a}$ and $\hat{b}$.

2. **Griffiths 4.59**. In classical electrodynamics the force on a apriticle of charge q moving with velocity $\mathbf{v}$ through electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ is given by the **Lorentz force law:**

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}).$$

   The force cannot be expressed as the gradient of a scalar potential energy function, and therefore the Schrodinger equation in its original form cannot accomodate it. But in the more sophisticated form

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

   there is no problem; the classical Hamiltonian is

$$H = \frac{1}{2m}(\mathbf{p} - a\mathbf{A})^2 + q\phi,$$

   where $\mathbf{A}$ is the vector potential ($\mathbf{B} = \nabla\times\mathbf{A}$) and ϕ is the scalar potential ($\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$), so the Schrodinger equation (making the canonical substitution ($\mathbf{p}\rightarrow(\hbar/i)\nabla$) becomes

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[\frac{1}{2m}\left(\frac{\hbar}{i}\nabla - q\mathbf{A}\right)^2 + q\phi\right]\Psi. \qquad (1)$$

   (a) Show that

$$\frac{d\langle\mathbf{r}\rangle}{dt} = \frac{1}{m}\langle(\mathbf{p} - q\mathbf{A})\rangle.$$

   As always we identify $d\langle\mathbf{r}\rangle/dt$ with $\langle\mathbf{v}\rangle$.

(b) Show that

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = q\langle \mathbf{E}\rangle + \frac{q}{2m}\langle(\mathbf{p}\times\mathbf{B} - \mathbf{B}\times\mathbf{p})\rangle - \frac{q^2}{m}\langle(\mathbf{A}\times\mathbf{B})\rangle.$$

(c) In particular, if the fields $\mathbf{E}$ and $\mathbf{B}$ are *uniform* over the volume of the wave packet, show that

$$m\frac{d\langle \mathbf{v}\rangle}{dt} = q(\mathbf{E} + \langle \mathbf{v}\rangle \times \mathbf{B}),$$

so the *expectation value* of $\langle \mathbf{v}\rangle$ moves according to the Lorentz force law, as we would expect from Ehrenfest's theorem.

3. **Griffiths 4.61**. In classical electrodynamics the potential $\mathbf{A}$ and ϕ are not uniquely determined; the *physical* quantities are the *fields*, $\mathbf{E}$ and $\mathbf{B}$.

(a) Show that the potentials

$$\phi' \equiv \phi - \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A}' \equiv \mathbf{A} + \nabla\Lambda \tag{2}$$

(where Λ is an arbitrary real function of position and time) yield the same fields as ϕ and $\mathbf{A}$. Equation 2 is called a **gauge transformation**, and the theory is said to be **gauge invariant**.

(b) In quantum mechanics the potentials play a more direct role, and it is of interest to know whether the theory remains gauge invariant. Show that

$$\Psi' \equiv e^{iq\Lambda/\hbar}\Psi$$

satisfies the Schrodinger equation (1) with the gauge-transformed potentials ϕ' and $\mathbf{A}'$. Since ψ' differs from ψ only by a *phase factor*, it represents the same physical state, and the theory *is* gauge invariant.

4. **Griffiths 5.1**. Typically , the interaction potential depends only on the vector $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ between the two particles. In that case the Schrodinger equation separates, if we change variables from $\mathbf{r}_1, \mathbf{r}_2$ to $\mathbf{r}$ and $\mathbf{R} \equiv (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/m_1 + m_2)$ (the center of mass).

(a) Show that $\mathbf{r}_1 = \mathbf{R} + (\mu/m_1)\mathbf{r}$, $\mathbf{r}_2 = \mathbf{R} - (\mu/m_2)\mathbf{r}$, and $\nabla_1 = (\mu/m_2)\nabla_R + \nabla_r$, $\nabla_2 = (\mu/m_1)\nabla_R - \nabla_r$, where

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$$

is the **reduced mass** of the system.

(b) Show that the (time-independent) Schrodinger equation becomes

$$-\frac{\hbar^2}{2(m_1 + m_2)}\nabla_R^2\psi - \frac{\hbar^2}{2\mu}\nabla_r^2\psi + V(\mathbf{r})\psi = E\psi.$$

(c) Separate the variables, letting $\psi(\mathbf{R}, \mathbf{r}) = \psi_R(\mathbf{R})\psi_r(\mathbf{r})$. Note that ψ_R satistifes the one-particle Schrodinger equation, with *total* mass $(m_1 + m_2)$ in place of m, potential zero, and energy E_R, while ψ_r satisfies the one-particle Schrodinger equation with the *reduced* mass in place of m, potential $V(\mathbf{r})$, and energy E_R. The total energy is the sum: $E = E_R + E_r$. What this tells is is that the center of mass moves like a free particle, and the *relative* motion (that is, the motion of particle 2 with respect to particle 1) is the same as if we had a *single* particle with the *reduced* mass, subject to the potential V. Exactly the same decomposition occurs in *classical* mechanics; it reduces the two-body problem to an equivalent one-body problem.

5. **Entangled states**

In this problem we consider a thought experiment proposed only recently by L. Hardy (1993). It illustrates the peculiar and counter-intuitive nature of entangled states in a different and in some ways simpler manner than the usual Bell's inequality experiments, which employ atomic cascades and 2-photon polarization correlations. The present thought experiment makes use of a source S and two detectors D_L and D_R (L, R for left and right respectively... you will need to draw yourself a picture). Each detector has two modes 1,2 determined by the position of a switch $K_{L,R}$. Each detector is equipped with a light that can flash either green or red. An experimental trial commences when the observer presses a button that launches a pair of correlated particles from source S; one particle goes to the left and the other to the right. After they have been emitted from the source but before they arrive at their respective detectors, the obserrver flips one coin to determine the position of K_L, and another coin to determine the position of K_R.

The arrival of a particle at D_L is indicated by the flashing of the green or red light there; similarly for the arrival of the other particle at D_R. The outcome of a given trial is specified by giving the positions of the two switches and the color of lights which flashed; for example (1G2R) signifies that K_L was in position 1 and D_L flashed green, while K_R was in position 2 and D_R flashed red.

The observer repeats the experiment, writing down the outcome for each trial, and finds the following results after many trials.

1. When both switches are in position 1, both lights never flash red: $(1R1R)$ never occurs.
2. When the switches are in different positions, both lights never flash green: $(1G2G)$ and $(2G1G)$ never occur.
3. In a non-zero fraction of the trials when both switches are in position 2, the lights both flash green: $(2G2G)$ sometimes occurs.

It is tempting to try to make the following (classical) analysis: Something in the common origin of the particles must be responsible for the observed correlations. Since the switches $K_{L,R}$ are not set until after the particles leave the source, whatever features the particles possess cannot depend on how these switches are set. Furthermore we can safely assume that D_L can only respond to the particle on the left, while D_R can only respond to the particle on the right. Then, since a trial could end up as a (1,2) or (2,1) trial, whenever one of the particles is of a variety to allow a type 2 detector to flash green, the other particle must be of a variety to make a type 1 detector flash red. (This follows from (2) above). Then in any of the occasional (2,2) trials where both detectors flashed green, both particles must have been of the variety to make a type 1 detector flash red. In other words, had both switches been set to position 1 in these trials, the outcome $(1R1R)$ would have been observed. However $(1R1R)$ is never observed! Thus the foregoing classical argument leads to a contradiction.

Nevertheless it is possible to set up such an experiment and to get the results given, but we must use quantum mechanics to describe the system of particles. Suppose that when a switch K is set to position 1 the outcome "green" corresponds to absorption of a particle of spin 1/2 with spin up along the z axis, whereas the outcome red corresponds to spin down:

$$| 1G\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad | 1R\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

Since we never obtain the outcome $(1R1R)$ we can assume that the quantum state of the two particles launched from the source is of the form:

$$| \psi\rangle = \alpha | 1R1G\rangle + \beta | 1G1R\rangle + \gamma | 1G1G\rangle$$

where $| 1R1G\rangle$ refers to left particle with spin down, right particle with spin up, and so forth, and α, β and γ are constants. We may assume that $| \psi\rangle$ is normalized to unity, so that $|\alpha|^2 + \beta|^2 + |\gamma|^2 = 1$.

(a) Show that, since outcomes $(1G2G)$ and $(2G1G)$ never occur and the states $| 1G2G\rangle$ and $| 2G1G\rangle$ are thus orthogonal to $| \psi\rangle$, it follows that:

$$\alpha\langle 2G | 1R\rangle + \gamma\langle 2G | 1G\rangle = 0$$
$$\beta\langle 2G | 1R\rangle + \gamma\langle 2G | 1G\rangle = 0$$

(b) It must be possible to express $| 2G\rangle$ as a linear combination of the states $| 1G\rangle, | 1R\rangle$; and $| 2R\rangle$ must be an orthogonal linear combination:

$$\begin{aligned} | 2G\rangle &= q^{\frac{1}{2}} | 1G\rangle + \sqrt{1-q} | 1R\rangle \\ | 2R\rangle &= -\sqrt{1-q} | 1G\rangle + q^{\frac{1}{2}} | 1R\rangle \end{aligned}$$

where $0 < q < 1$. Show that, since outcome $(2G2G)$ sometimes occurs, and therefore $|\langle 2G2G | \psi\rangle|^2 = p \neq 0$, it follows that:

$$p = \frac{q^2(1-q)^2}{1-q^2}$$

(c) Show that when p is maximized in part (b) the probabilities of the various outcomes are given by the following table where $z = \frac{1}{2}(\sqrt{5} - 1)$:

Outcome	Probability	Outcome	Probability
$1G1G$	z^3	$2G1G$	0
$1G1R$	z^2	$2G1R$	z^3
$1R1G$	z^2	$2R1G$	z
$1R1R$	0	$2R1R$	z^4
$1G2G$	0	$2G2G$	$z^5 = p$
$1G2R$	z	$2G2R$	z^4
$1R2G$	z^3	$2R2G$	z^4
$1R2R$	z^4	$2R2R$	z