Physics 443, Solutions to PS 6

1. **Griffiths 4.13**.

   (a) Find $\langle r \rangle$ and $\langle r^2 \rangle$ for an electron in the ground state of hydrogen. Express your answers in terms of the Bohr radius.

   [The ground state of the Hydrogen wavefunction can be written as

   $$\psi_{100} = \frac{\exp(-\frac{r}{a})}{\sqrt{\pi a^3}}$$

   where a is the Bohr radius. We can then calculate

   $$\langle r \rangle = \frac{1}{\pi a^3} \int r^3 e^{-2r/a} dr d\Omega = 4a \int_0^\infty u^3 e^{-2u}\ du = \frac{3a}{2}.$$

   Similarly,

   $$\langle r^2 \rangle = 4a^2 \int u^4 e^{-2u}\ du = 3a^2].$$

   (b) Find $\langle x \rangle$ and $\langle x^2 \rangle$ for an electron in the ground state of hydrogen. *Hint* : This requires no new integration - note that $r^2 = x^2 + y^2 + z^2$, and exploit the symmetry of the ground state.

   [We have that

   $$\langle x \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d(\cos\theta) \int_0^\infty \frac{e^{-2r/a}}{\sqrt{\pi a^3}} r^2 dr = 0$$

   and by symmetry

   $$\langle x^2 \rangle = \langle y^2 \rangle = \langle z^2 \rangle = \langle r^2 \rangle / 3 = a^2].$$

   (c) Find $\langle x^2 \rangle$ in the state $n = 2, l = 1, m = 1$. *Warning* : This is *not* symmetrical in x, y, z. Use $x = r \sin\theta \cos\theta$.

   [For part (c), we write

   $$\psi_{211} = -\sqrt{\frac{3}{8\pi}} \frac{1}{\sqrt{24a^3}} \frac{r}{a} e^{\frac{-r}{2a}} \sin\theta e^{i\phi}.$$

   To calculate the e xpectation value

   $$\begin{aligned} \langle x^2 \rangle &= \frac{3}{8\pi}\left(\frac{1}{24a^3}\right) \int \left(\frac{r}{a}\right)^2 e^{\frac{-r}{a}} \sin^2\theta (r^2 \sin^2\theta \cos^2\phi)(r^2 \sin\theta\ dr d\theta d\phi) \\ &= \frac{3}{8\pi}\left(\frac{1}{24a^5}\right) \int_0^{2\pi} \cos^2\phi d\phi \int_0^{\pi} \sin^5\theta d\theta \int_0^\infty r^6 e^{\frac{-r}{a}} dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{8\pi}\frac{1}{24a^5}(\pi)\int_{-1}^{1}(1-x^2)^2dx\int_0^\infty a^7u^6e^{-u}du \\
&= \frac{3}{8\pi}\frac{1}{24a^5}(\pi)\left(\frac{16}{15}\right)\int_0^\infty a^7u^6e^{-u}du \\
&= \frac{3}{8\pi}\frac{1}{24a^5}(\pi)\left(\frac{16}{15}\right)a^7 6! \\
&= 12a^2]
\end{aligned}
$$

2. **Griffiths 4.16**. In this problem notice that

$$V(r) = \frac{-Ze^2}{4\pi\epsilon_0}\frac{1}{r}$$

is just the same potential as the hydrogen atom with $e^2 \to Ze^2$. Which means that we can use all the results of the Hydrogen atom making this substitution. Looking at the denependence of these functions on e^2, we can write down the answers as:

$$E_n(Z) = Z^2\epsilon_n; a(z) = \frac{a}{Z}; R(Z) = Z^2R$$

$$\frac{1}{\lambda}\bigg|_{\text{Lyman}} = \left(\frac{4}{3R}, \frac{1}{R}\right) \to \left(\frac{4}{3Z^2R}, \frac{1}{Z^2R}\right)$$

$$\text{For Z} = 2, \quad (2.28\text{x}10^{-8}\text{m}, 3.04\text{x}10^{-8}\text{m}) \in \text{ultraviolet}$$

$$\text{For Z} = 3, \quad (1.01\text{x}10^{-8}\text{m}, 1.35\text{x}10^{-8}\text{m}) \in \text{ultraviolet}$$

3. **Griffiths 4.29**.

(a) Find the eigenvalues and eigenspinors of $\mathbf{S}_y$.
[The eigenvalues of S_y are $\pm\hbar/2$. The eigenvalues of a spin $\frac{1}{2}$ matrix are $\pm\frac{1}{2}$ regardless of axis. Then

$$
\begin{aligned}
\frac{\hbar}{2}\sigma_y\chi_\pm^{(y)} &= \pm\frac{\hbar}{2}\chi_\pm^{(y)} \\
\to \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ b \end{pmatrix} &= \pm\begin{pmatrix} 1 \\ b \end{pmatrix} \\
\to b &= \pm i
\end{aligned}
$$

The normalized eigenvectors are

$$\chi_\pm^{(y)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm i \end{pmatrix}].$$

(b) If you measured S_y on a particle in the general state χ

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$

what values might you get, and what is the probability of each? Check that the probabilities add up to 1. *Note* : a and b need not be real.

[We can write that

$$\begin{aligned} \chi &= \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_- \\ &\text{or in the } y - \text{basis} \\ \chi &= \begin{pmatrix} a' \\ b' \end{pmatrix} = a'\chi_+^{(y)} + b'\chi_-^{(y)} \end{aligned}$$

The probability that we find the particle with spin $+\frac{\hbar}{2}$, that is, $P_{\frac{1}{2}}$ is

$$\begin{aligned} P_{\frac{1}{2}} &= |a\chi_+^{(y)\dagger}\chi_+ + b\chi_+^{(y)\dagger}\chi_-|^2 \\ &= |\frac{a}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}| \\ &= |\frac{a}{\sqrt{2}} + \frac{ib}{\sqrt{2}}|^2 \\ &= \frac{1}{2}(|a|^2 + ia^*b - ib^*a + |b|^2) \end{aligned}$$

The probability that we find the particle with spin $-\frac{\hbar}{2}$ is

$$\begin{aligned} P_{-\frac{1}{2}} &= |a\chi_-^{(y)\dagger}\chi_+ + b\chi_-^{(y)\dagger}\chi_-|^2 \\ &= |\frac{a}{\sqrt{2}}\begin{pmatrix} 1 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix} 1 & -i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}| \\ &= |\frac{a}{\sqrt{2}} + \frac{-ib}{\sqrt{2}}|^2 \\ &= \frac{1}{2}(|a|^2 - ia^*b + ib^*a + |b|^2) \end{aligned}$$

$P_{\frac{1}{2}} + P_{-\frac{1}{2}} = |a|^2 + |b|^2 = 1.$]

(c) If you measured S_y^2, what values might you get, and with what probabilities?
[$S_y^2 = \frac{\hbar}{4}$ for either of the two eigenstates. So we measure $(S_y)^2 = \hbar/4$ with unit probability.]

4. **Griffiths 4.30**. We can begin by constructing

$$\begin{aligned} S_r &= \frac{\hbar}{2}\left[\sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z\right], \\ &= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}. \end{aligned}$$

Solving for the eigenvalues, we have that $(\cos\theta-\lambda)(\cos\theta+\lambda)+\sin^2\theta = 0$, giving us eigenvalues $\lambda = \pm\hbar/2$. The eigenvectors are found using the normal procedure. For $\chi_+ = (x, y)$, we have that $(\cos\theta - 1)x + \sin\theta\exp(-i\phi)y = 0$, or after applying a trig identity, $y\cos(\theta/2) = \exp(i\phi)\sin(\theta/2)x$. And normalization requires that $|x|^2 + |y|^2 = 1$, or $|x|^2(1+\tan^2(\theta/2)) = 1$, giving $x = \cos(\theta/2)$ and $y = \sin(\theta/2)\exp(i\phi)$. And similarly for χ_-. The answers are:

$$\chi_+ = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}, \quad \chi_- = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}.$$

5. **Griffiths 4.33**. An electron is at rest in an oscillating magnetic field

$$\mathbf{B} = B_0\cos(\omega t)\hat{k},$$

where B_0 and ω are constants.

(a) Construct the Hamiltonian matrix for this system.
[The Hamiltonian for this system can be written as $H = -\mu\cdot\mathbf{B} = -\gamma\mathbf{B}\cdot\mathbf{S} = -\gamma\mathbf{B}\cdot\mathbf{S} = -\gamma B_0\cos(\omega t)\frac{\hbar}{2}\sigma_z$.]

(b) The electron starts out (at $t = 0$) in the spin-up state with respect to the x-axis (that is: $\chi(0) = \chi_+^{(x)}$). Determine $\chi(t)$ at any subsequent time. *Beware* : This is a time-*dependent* Hamiltonian, so you cannot get $\chi(t)$ in the usual way from stationary states. Fortunately, in this case you can solve the time-dependent Schrodinger equation

$$i\hbar\frac{\partial\chi}{\partial t} = \mathbf{H}\chi,$$

directly.
[From Schrodinger's equation we get that

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}a\\b\end{pmatrix} = -\gamma B_0\cos(\omega t)\frac{\hbar}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

where $\chi(t) = \begin{pmatrix}a(t)\\b(t)\end{pmatrix}$. Then we have a pair of differential equations

$$\begin{aligned} i\hbar\frac{\partial a}{\partial t} &= -\gamma B_0\cos(\omega t)\frac{\hbar}{2}a \\ \Rightarrow \frac{da}{a} &= i\frac{\gamma B_0\cos(\omega t)}{2}dt \\ \Rightarrow a &= a(0)\exp(\frac{i\gamma B_0\sin(\omega t)}{2\omega}) \end{aligned}$$

Similarly

$$\begin{aligned} i\hbar\frac{\partial b}{\partial t} &= \gamma B_0\cos(\omega t)\frac{\hbar}{2}b \\ \Rightarrow b &= b(0)\exp(\frac{i\gamma B_0\sin(\omega t)}{2\omega}) \end{aligned}$$

At $t = 0$,

$$\chi(0) = \begin{pmatrix}a(0)\\b(0)\end{pmatrix} = \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Therefore

$$\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix}\exp(-\frac{i\gamma B_0\sin(\omega t)}{2\omega})\\ \exp(\frac{i\gamma B_0\sin(\omega t)}{2\omega})\end{pmatrix}$$

(c) Find the probability of getting $\hbar/2$, if you measure S_x.
[The probability to get $S_x = -\hbar/2$ is given by the projection of $\chi(t)$ onto the eigenstate of S_z with eigenvalue -$\frac{\hbar}{2}$, namely $\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$. Then the probability is

$$\begin{aligned} |\left\langle\chi_-^{(x)} \mid \chi(t)\right\rangle|^2 &= |\frac{1}{2}(e^{-\frac{i\gamma B_0\sin\omega t}{2\omega}} - e^{\frac{i\gamma B_0\sin(\omega t)}{2\omega}})|^2 \\ &= \sin^2\frac{\gamma B_0\sin(\omega t)}{2\omega}] \end{aligned}$$

(d) What is the minimum field (B_0) required to force a complete flip in S_x?

[We see that the minimum field for a spin flip is that required so that $|\left\langle \chi_-^{(x)} \mid \chi(t) \right\rangle|^2 = 1$ which will occur only if $\frac{\gamma B_0}{2\omega} \geq \frac{\pi}{2}$, or if $B_0 \geq \frac{\pi\gamma}{\omega}$.]

6. **Griffiths 4.36**. This problem involves reading out values from the Clebsh-Gorden Table on pg. 168 of Griffiths. Part (a) asks that we find the co-efficients of the following product in which the total spin is 3 and the z-component is 1.

$$|3,1\rangle = (?)|1,1\rangle \otimes |2,0\rangle + (?)|1,0\rangle \otimes |2,1\rangle + (?)|1,-1\rangle \otimes |2,2\rangle.$$

Looking at the table we can fill in the co-efficients as

$$|3,1\rangle = \sqrt{\frac{6}{15}}|1,1\rangle \otimes |2,0\rangle + \sqrt{\frac{8}{15}}|1,0\rangle \otimes |2,1\rangle + \sqrt{\frac{1}{15}}|1,-1\rangle \otimes |2,2\rangle.$$

We can then read off the probabilities of the z-component of the spin-2 particle as $P(-2\hbar) = 0, P(-1\hbar) = 0, P(0) = 6/15, P(1\hbar) = 8/15, P(2\hbar) = 1/15$.

For part(b), we have to add the angular momentum for an electron with orbital ket $|1,0\rangle$ and spin ket $|1/2,-1/2\rangle$. Again this is just looking up the Clebsh-Gordon table to find that

$$|1,0\rangle \otimes |1/2,-1/2\rangle = \sqrt{\frac{2}{3}}|3/2,-1/2\rangle + \sqrt{\frac{1}{3}}|1/2,-1/2\rangle.$$

We have that with probability 2/3 we will have $J = 3/2$ or $J^2 = J(J+1) = 15/4\ \hbar^2$, and probability 1/3 that we will have $J = 1/2$, or $J^2 = J(J+1) = 3/4\ \hbar^2$.

7. **Show that**

$$e^{i(\sigma\cdot\hat{n})\alpha/2} = \cos(\alpha/2) + i(\hat{n}\cdot\sigma)\sin(\alpha/2)$$

where the unit vector

$$\hat{n} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

The operator $\exp(i(\sigma\cdot\hat{n})\alpha/2)$ effects a rotation of the spinor χ through the angle α about the axis $\hat{n}$.

[We start with

$$\exp(i\hat{n}\cdot\vec{\sigma}\alpha/2) \quad = \quad I + i\hat{n}\cdot\vec{\sigma}\frac{\alpha}{2} + \frac{1}{2}\left(i\hat{n}\cdot\vec{\sigma}\frac{\alpha}{2}\right)^2 + \frac{1}{3!}\left(i\hat{n}\cdot\vec{\sigma}\frac{\alpha}{2}\right)^3 + ...$$

Now

$$\begin{aligned}
(\hat{n}\cdot\vec{\sigma})^2 &= (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 \\
&= (n_x^2 + n_y^2 + n_z^2)I \\
&= +n_xn_y(\sigma_x\sigma_y + \sigma_y\sigma_x) + n_xn_z(\sigma_x\sigma_z + \sigma_z\sigma_x) + n_zn_y(\sigma_z\sigma_y + \sigma_y\sigma_z) \\
&= I
\end{aligned}$$

where I is the identity matrix. Then we have

$$\exp(i\hat{n}\cdot\vec{\sigma}\alpha/2) \quad = \quad I + i\hat{n}\cdot\vec{\sigma}\frac{\alpha}{2} - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2 - i\frac{1}{3!}\left(\hat{n}\cdot\vec{\sigma}\right)\left(\frac{\alpha}{2}\right)^3 + ...$$

$$\begin{aligned}
\exp(\frac{i\hat{n}.\vec{\sigma}\alpha}{2}) &= I\left(1 - \frac{(\alpha/2)^2}{2!} + \cdots\right) + i(\hat{n}.\vec{\sigma})\left(\alpha/2 - \frac{(\alpha/2)^3}{3!} + \cdots\right) \\
&= I\cos(\frac{\alpha}{2}) + i(\hat{n}.\vec{\sigma})\sin(\frac{\alpha}{2}).]
\end{aligned}$$