Physics 443, Solutions to PS 5

1. Angular Momentum

   (a) Since $l = 1$ and $m = \pm 1, 0$, we have that

$$L^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \text{and} \quad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \text{and} \quad L_- = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}.$$

   Then

$$L^2 \chi_m = \hbar^2 l(l+1)\chi_m = \hbar^2 2\chi_m$$

   where $l = 1$.

$$L_z \chi_m = \hbar m \chi_m.$$

   and

$$L_\pm \chi_m = \hbar\sqrt{l(l+1) - m(m\pm 1)}\chi_{m\pm 1}$$

   To calculate L_y, we use that

$$L_y = \frac{L_+ - L_-}{2i} = \frac{\hbar}{2i} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{pmatrix} = \frac{\hbar}{i\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{\hbar}{i} J.$$

   (b) Using that $R_y(\theta) = \exp(iL_y\theta/\hbar) = \exp(J\theta)$, and doing a Taylor expansion we find that

$$\begin{aligned}
&= I + \theta J + \frac{1}{2}\theta^2 J^2 + \frac{1}{3!}\theta^3 J^3 + \frac{1}{4!}\theta^4 J^4 \ldots \\
&= I + \theta J + \frac{1}{2}\theta^2 J^2 - \frac{1}{3!}\theta^3 J - \frac{1}{4!}\theta^4 J^2 \ldots \\
&= I + J\cos\theta + J^2(1-\cos\theta) \\
&= \frac{1}{2} \begin{pmatrix} 1+\cos\theta & \sqrt{2}\sin\theta & 1-\cos\theta \\ -\sqrt{2}\sin\theta & 2\cos\theta & \sqrt{2}\sin\theta \\ 1-\cos\theta & -\sqrt{2}\sin\theta & 1+\cos\theta \end{pmatrix}.
\end{aligned}$$

   where we have used the fact that $J^3 = -J$ and $J^4 = -J^2$.

2. **Griffiths 4.19**.

(a)

$$\begin{aligned}
[L_z, x] &= [xp_y - yp_x, x] = [xp_y, x] - [yp_x, x] = -y[p_x, x] = i\hbar y \\
[L_z, y] &= [xp_y - yp_x, y] = [xp_y, y] - [yp_x, y] = x[p_y, y] = -i\hbar x \\
[L_z, z] &= [xp_y - yp_x, z] = [xp_y, z] - [yp_x, z] = 0
\end{aligned}$$

$$\begin{aligned}
[L_z, p_x] &= [xp_y - yp_x, p_x] = [xp_y, p_x] - [yp_x, p_x] = p_y[x, p_x] = i\hbar p_y \\
[L_z, p_y] &= [xp_y - yp_x, p_y] = [xp_y, p_y] - [yp_x, p_y] = -p_x[y, p_y] = -i\hbar p_x \\
[L_z, p_z] &= [xp_y - yp_x, p_z] = [xp_y, p_z] - [yp_x, p_z] = 0
\end{aligned}$$

(b)

$$\begin{aligned}
[L_z, L_x] &= [L_z, yp_z - zp_y] \\
&= y[L_z, p_z] + [L_z, y]p_z - z[L_z, p_y] - [L_z, z]p_y \\
&= 0 - i\hbar xp_z + i\hbar zp_x + 0 \\
&= i\hbar(zp_x - xp_z) = i\hbar L_y
\end{aligned}$$

(c)

$$\begin{aligned}
[L_z, r^2] &= [L_z, x^2] + [L_z, y^2] + [L_z, z^2] \\
&= x[L_z, x] + [L_z, x]x + y[L_z, y] + [L_z, y]y + z[L_z, z] + [L_z, z]z \\
&= i2\hbar xy - i2\hbar yx + 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[L_z, p^2] &= [L_z, p_x^2] + [L_z, p_y^2] + [L_z, p_z^2] \\
&= p_x[L_z, p_x] + [L_z, p_x]p_x + p_y[L_z, p_y] + [L_z, p_y]p_y + p_z[L_z, p_z] + [L_z, p_z]p_z \\
&= i2\hbar p_x p_y - i2\hbar p_y p_x + 0 \\
&= 0
\end{aligned}$$

(d)

$$\begin{aligned}
[H, \mathbf{L}] &= [\frac{p^2}{2m}, \mathbf{L}] + [V(r), \mathbf{L}] \\
&= \frac{1}{2m}\left([p^2, L_x]\hat{x} + [p^2, L_y]\hat{y} + [p^2, L_z]\hat{z}\right) \\
&\qquad + [V(r), L_x]\hat{x} + [V(r), L_y]\hat{y} + [V(r), L_z]\hat{z} \\
&= 0
\end{aligned}$$

In the last step we take advantage of the fact that if $[L_z, p^2] = 0$, then the same must be true for L_x, and L_y. The x,y and z components of the angular momentum operator can be written as differential operators that are functions only of θ and ϕ. Since the operator does not have an r dependence it will commute with a function $V(r)$ that depends only on r.

3. **Griffiths 4.20**. From the Equation of Motion, we have that

$$\frac{d}{dt}\langle \mathbf{L}\rangle = \frac{i}{\hbar}\langle [H, \mathbf{L}]\rangle.$$

We can calculate this commutator as follows

$$[H, L] = [\frac{p^2}{2m}, \mathbf{L}] + [V, \mathbf{r} \text{ x } \mathbf{p}]. \tag{1}$$

We showed in problem in problem 2 (Griffiths 4.19), that $[L_z, p^2] = 0$. The same is true for L_x and L_y so the first term vanishes. . Then

$$[V, \mathbf{r} \times \mathbf{p}] = -\mathbf{r} \times [V, \mathbf{p}] - [V(\mathbf{r}), \mathbf{r}] \times \mathbf{p}$$

The second commutator is zero since $V(\mathbf{r})$ is a function of $\mathbf{r}$. For the second term, you can write $\mathbf{p} = -i\hbar\nabla$. Then

$$\mathbf{r} \times [V, \mathbf{p}] = -i\hbar\mathbf{r} \times [V, \nabla] = i\hbar\mathbf{r} \times \nabla V$$

It follows that

$$\frac{d}{dt}\langle \mathbf{L}\rangle = \mathbf{r} \times (-\nabla V(\mathbf{r})).$$

For a potential that depends only on the magnitude of $\mathbf{r}$, we see that the gradient of $V(|\mathbf{r}|)$ is in the $\hat{\mathbf{r}}$ direction, and $\mathbf{r} \times \hat{\mathbf{r}} = 0$ giving us that the angular momentum is conserved.

4. **Griffiths 4.22**. Being a state of maximum L_z, we get that $L_+ Y_l^l = 0$. To get the functional form,we write L_+ as a differential operator,

$$L_+ = \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right).$$

Then

$$\begin{aligned} 0 &= L_+ Y_l^l \\ &= \hbar e^{i\phi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right) Y_l^l(\theta, \phi) \\ \rightarrow 0 &= \left(\frac{1}{\cot\theta}\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\phi}\right) Y_l^l(\theta, \phi) \end{aligned}$$

We try separating variables and write $Y_l^l(\theta,\phi) = g(\theta)h(\phi)$ and then

$$\begin{aligned} 0 &= \left(\frac{1}{\cot\theta}\frac{\partial}{\partial\theta} + i\frac{\partial}{\partial\phi}\right) g(\theta)h(\phi) \\ \rightarrow \frac{1}{\cot\theta g(\theta)}\frac{\partial}{\partial\theta}g(\theta) &= -\frac{i}{h(\phi)}\frac{\partial}{\partial\phi}h(\phi) = k \end{aligned}$$

As usual since all of the θ dependence is on the left and all of the ϕ dependence on the right, then both are equal to a constant, k. Then

$$\frac{dh(\phi)}{h} = ikd\phi \rightarrow h = (\text{constant})e^{ik\phi}$$

Also

$$\begin{aligned} \frac{dg}{g} &= k\cot\theta g \rightarrow \ln(g) = k\int \cot\theta d\theta \\ &= k\ln\sin\theta + \text{constant} \\ \Rightarrow g &= c\sin^k\theta \end{aligned}$$

And so $Y_l^l(\theta,\phi) = c\sin^k\theta e^{ik\phi}$. We use the fact that Y_l^l is an eigenstate of L_z with eigenvalue $\hbar l$ to determine k.

$$\begin{aligned} L_z Y_l^l &= \hbar\frac{\partial}{\partial\phi}c\sin^k\theta e^{ik\phi} \\ \hbar l &= \hbar k c\sin^k\theta e^{ik\phi} \\ \Rightarrow k &= l \end{aligned}$$

We fix the normalization constant by integrating.

$$\begin{aligned} 1 &= |c|^2\int_0^{2\pi}\int_0^{\pi}\sin^{2l}\theta\sin\theta d\theta d\phi \\ 1 &= 2\pi|c|^2\int\sin^{2l+1}(\theta)\ d\theta. \end{aligned}$$

We use that

$$\int_0^{\frac{\pi}{2}}\sin^{2p-1}\cos^{2q-1} = \frac{\Gamma(p)\Gamma(q)}{2\Gamma(p+q)}$$

In our case, $q = 1/2$, and $p = l+1$. Putting it all together, we have that

$$1 = 4\pi|c|^2\frac{\Gamma(l+1)\sqrt{\pi}}{2\Gamma(l+3/2)}$$

In the particluar case that $l = 3$, we have

$$c = \sqrt{\frac{\Gamma(l+3/2)}{\Gamma(l+1)2\pi\sqrt{\pi}}} = \sqrt{\frac{7!!}{3!\pi 2^5}} = \sqrt{\frac{35}{64\pi}}$$

5. **Griffiths, 4.27**. An electron is in the spin state

$$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

(a) Determine the normalization constant A.

$$\begin{aligned} [\chi^\dagger \chi &= 1 = |A|^2 (\, -3i \quad 4\,) \begin{pmatrix} 3i \\ 4 \end{pmatrix} = |A|^2 (25) \\ \Rightarrow A &= \frac{1}{5}] \end{aligned}$$

(b) Find the expectation values of S_x, S_y, and S_z.

$$\begin{aligned} [\langle S_x \rangle &= = \chi^\dagger S_x \chi = |A|^2 (\, -3i \quad 4\,) \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\ &= \frac{1}{25} (\, -3i \quad 4\,) \frac{\hbar}{2} \begin{pmatrix} 4 \\ 3i \end{pmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \langle S_y \rangle &= |A|^2 (\, -3i \quad 4\,) \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\ &= \frac{1}{25} (\, -3i \quad 4\,) \frac{\hbar}{2} \begin{pmatrix} -4i \\ -3 \end{pmatrix} \\ &= -\frac{24}{25} \frac{\hbar}{2} \end{aligned}$$

$$\begin{aligned} \langle S_z \rangle &= |A|^2 (\, -3i \quad 4\,) \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3i \\ 4 \end{pmatrix} \\ &= \frac{1}{25} (\, -3i \quad 4\,) \frac{\hbar}{2} \begin{pmatrix} 3i \\ -4 \end{pmatrix} \\ &= -\frac{7}{25} \frac{\hbar}{2}] \end{aligned}$$

(c) Find the "uncertainties" $\sigma_{S_x}, \sigma_{S_y}$, and σ_{S_z}. ($Note$: These sigmas are standard deviations, not Pauli matrices!) [Remember that $\sigma_{S_i^2} = \langle S_i^2 \rangle - \langle S_i \rangle^2$, and also that $S_i^2 = \frac{1}{3}S^2$. Then

$$\begin{aligned}
\sigma_{S_x}^2 &= \langle S_x^2 \rangle - \langle S_x \rangle^2 = \frac{\hbar^2}{4}(1-0) = \frac{\hbar^2}{4} \\
\sigma_{S_y}^2 &= \langle S_y^2 \rangle - \langle S_y \rangle^2 = \frac{\hbar^2}{4}(1 - \left(\frac{24}{25}\right)^2) = \frac{\hbar^2}{4}\left(\frac{7}{25}\right)^2 \\
\sigma_{S_z}^2 &= \langle S_z^2 \rangle - \langle S_z \rangle^2 = \frac{\hbar^2}{4}(1 - \left(\frac{7}{25}\right)^2) = \frac{\hbar^2}{4}\left(\frac{24}{25}\right)^2]
\end{aligned}$$

(d) Confirm that your results are consistent with all three uncertainty principles, namely

$$\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}|\langle L_z \rangle|$$

and its cyclic permutations.

$$\begin{aligned}
[\sigma_{S_x}\sigma_{S_y} &= \left(\frac{\hbar}{2}\right)^2\left(\frac{7}{25}\right) \geq \left(\frac{\hbar}{2}\right)\langle S_z \rangle = \left(\frac{\hbar}{2}\right)^2\left(\frac{7}{25}\right) \\
\sigma_{S_y}\sigma_{S_z} &= \left(\frac{\hbar}{2}\right)^2 \frac{7}{25}\frac{24}{25} \geq \left(\frac{\hbar}{2}\right)\langle S_x \rangle = 0 \\
\sigma_{S_z}\sigma_{S_x} &= \left(\frac{\hbar}{2}\right)^2 \frac{24}{25} \geq \left(\frac{\hbar}{2}\right)\langle S_y \rangle = \left(\frac{\hbar}{2}\right)^2 \frac{24}{25}]
\end{aligned}$$

6. **Griffiths 4.28**. For the most general normalized spinor χ where

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$

with

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

compute $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$, $\langle S_x^2 \rangle$, $\langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle$

$$[\langle S_x \rangle = \frac{\hbar}{2}\begin{pmatrix} a^* & b^* \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(a^*b + b^*a)$$

$$\begin{aligned}
\langle S_y \rangle &= \frac{\hbar}{2} \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{-i\hbar}{2}(a^* b - b^* a) \\
\langle S_z \rangle &= \frac{\hbar}{2} \begin{pmatrix} a^* & b^* \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}(|a|^2 - |b|^2)
\end{aligned}$$

Since $S_x^2 = S_y^2 = S_z^2 = \frac{1}{3}S^2$, it follows that

$$\langle S_x^2 \rangle = \langle S_y^2 \rangle = \langle S_z \rangle^2 = \frac{1}{3}\langle S^2 \rangle = \frac{1}{4}\hbar^2$$