Physics 443 HW #5
Due February 27, 2008

1. **Angular momentum 1**
Consider a system with total angular momentum 1 and with basis vectors

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \chi_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_{-1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

   (a) Derive matrix representations of $L_\pm, L_z, L^2$ and L_y.

   (b) Any state ψ with angular momentum $l = 1$ can be written as a linear combination of the base states.

$$\psi = a\chi_1 + b\chi_0 + c\chi_{-1}$$

   where a, b,and c are the amplitudes for the various components of angular momentum with respect to the z-axis. In a new coordinate system that is related to the original by a rotation by an angle θ about the y-axis, we can write

$$\psi = a'\chi_1' + b'\chi_0' + c'\chi_{-1}'$$

   a', b', and c' are amplitudes for components of angular momentum along the z' axis. The angle between the z axis and the z' axis is θ. Derive the rotation matrix $R_y(\theta)$ that relates a, b and c with a', b', and c'

2. **Griffiths 4.19**.

   (a) Starting with the canonical commutation relations for potistion and momentum

   $[r_i, p_j] = i\hbar\delta_{ij}$, where $r_1 = x, r_2 = y, r_3 = z$, and $p_1 = p_x, p_2 = p_y, p_3 = p_z$

   work out the following commutators:

$$\begin{array}{lll} [L_z, x] = i\hbar y, & [L_z, y] = -i\hbar x, & [L_z, z] = 0, \\ [L_z, p_x] = i\hbar p_y, & [L_z, p_y] = -i\hbar p_x, & [L_z, p_z] = 0. \end{array} \qquad (1)$$

   (b) Use these results to obtain $[L_z, L_x] = i\hbar L_y$ directly from

$$L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad L_z = xp_y - yp_x$$

(c) Evaluate the commutators $[L_z, r^2]$ and $[L_z, p^2]$ (where, of course, $r^2 = x^2 + y^2 + z^2$ and $p^2 = p_x^2 + p_y^2 + p_z^2$).

(d) Show that the Hamiltonian $H = (p^2/2m) + V$ commutes with all three components of $\mathbf{L}$, provided that V depends only on r. (Thus H, L^2, and L_z are mutually compatible observables.)

3. **Griffiths 4.20**.

   (a) Prove that for a paricle in a potential $V(\mathbf{r})$ the rate of change of the expectation value of the orbital angular momentum $\mathbf{L}$ is equal to the expectation value of the torque:

   $$\frac{d}{dt}\langle \mathbf{L} \rangle = \langle \mathbf{N} \rangle,$$

   where

   $$\mathbf{N} = \mathbf{r} \times (-\nabla V).$$

   (This is the rotational analog of Ehrenfest's theorem.)

   (b) Show that $d\langle \mathbf{L} \rangle/dt = 0$ for any spherically symmetric potential. (This is one form of the quantum statement of **conservation of angular momentum**.)

4. **Griffiths 4.22**

   (a) What is $L_+ Y_l^l$? (No calculation allowed!)

   (b) Use the result of (a), together with the fact that

   $$L_\pm = \pm \hbar e^{\pm i\phi} \left(\frac{\partial}{\partial \theta} \pm \cot\theta \frac{\partial}{\partial \phi} \right),$$

   and the fact that $L_z Y_l^l = \hbar l Y_l^l$, to determine $Y_l^l(\theta, \phi)$, up to a normalization constant.

   (c) Determine the normalization constant by direct integration. Compare your answer for $l = 3$ to what appears in the table on page 139

5. **Griffiths 4.27**. An electron is in the spin state

   $$\chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix}.$$

(a) Determine the normalization constant A.

(b) Find the expectation values of S_x, S_y, and S_z.

(c) Find the "uncertainties" $\sigma_{S_x}, \sigma_{S_y}$, and σ_{S_z}. (*Note* : These sigmas are standard deviations, not Pauli matrices!)

(d) Confirm that your results are consistent with all three uncertainty principles, namely

$$\sigma_{S_x}\sigma_{S_y} \geq \frac{\hbar}{2}|\langle L_z \rangle|$$

and its cyclic permutations.

6. **Griffiths 4.28**. For the most general normalized spinor χ where

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-,$$

with

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

compute $\langle S_x \rangle, \langle S_y \rangle, \langle S_z \rangle, \langle S_x^2 \rangle, \langle S_y^2 \rangle$, and $\langle S_z^2 \rangle$. Check that $\langle S_x^2 \rangle + \langle S_y^2 \rangle + \langle S_z^2 \rangle = \langle S^2 \rangle$