Physics 443 HW #4
Due February 20, 2008

1. **Neutrino Oscillations**
Electron and muon neutrinos (ν_e and ν_μ) are produced in weak interactions, like neutron decay or π decay. Recent measurements indicate that there is some amplitude for an electron neutrino to turn into a muon neutrino and vice versa. ($\nu_e \leftrightarrow \nu_\mu$) We can represent the general state as a linear combination

$$|\,\psi\rangle = a|\,\nu_e\rangle + b|\,\nu_\mu\rangle$$

where

$$\begin{aligned}\langle\nu_e\,|\,\nu_e\rangle &= \langle\nu_\mu\,|\,\nu_\mu\rangle = 1\\ \langle\nu_e\,|\,\nu_\mu\rangle &= \langle\nu_\mu\,|\,\nu_e\rangle = 0\end{aligned}$$

Then we can write

$$|\,\nu_e\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad |\,\nu_\mu\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

The Hamiltonian matrix is

$$H = \begin{pmatrix}\langle\nu_e\,|\,H\,|\,\nu_e\rangle & \langle\nu_e\,|\,H\,|\,\nu_\mu\rangle\\ \langle\nu_\mu\,|\,H\,|\,\nu_e\rangle & \langle\nu_\mu\,|\,H\,|\,\nu_\mu\rangle\end{pmatrix} = \begin{pmatrix}E_0 & A\\ A & E_0\end{pmatrix}$$

(Note that the Hamiltonian matrix elements are all real and it is hermitian. Unlike in the K^0 system the total number of neutrinos is conserved.)

   (a) Find the energy eigenvalues and eigenvectors. Normalize the eigenvectors and show that they are orthogonal. (We usually label the eigenstates $|\,\nu_1\rangle$ and $|\,\nu_2\rangle$.)
   (b) Write the matrix S that transforms from the $|\,\nu_e\rangle, |\,\nu_\mu\rangle$ basis to the basis in which the Hamiltonian is diagonal. Transform both pairs of basis vectors to the diagonal basis.
   (c) Determine $|\,\psi(t)\rangle$
   (d) Electron neutrinos are produced in nuclear reactions in the sun. Muon neutrinos are not. If at $t = 0$ we have a state of pure electron neutrinos,

   $$|\,\psi(t)\rangle = |\,\nu_e\rangle$$

   what is the probability that at time t it would have transformed into a muon neutrino? What is the probability that it would be an electron neutrino? What energy would we measure and with what probability?

WKB approximation

2. **Griffiths 8.1**.

   Use the WKB approximation to find the allowed energies (E_n) of an inginite square well with a "self", of height V_0 extending half-way across.

$$V(x) = \begin{cases} V_0, & \text{if } 0 < x < a/2, \\ 0, & \text{if } a/2 < x < a, \\ \infty, & \text{otherwise} \end{cases} \tag{1}$$

   Express your answer in terms of V_0 and $E_n^0 \equiv (n\pi\hbar)^2/2ma^2$ (the nth allowed energy for the infinite square well with *no* shelf). Assume that $E_1^0 > V_0$, but do *not* assume that $E_n \gg V_0$. (Never mind the comparison with Example 6.1)

3. **Griffiths 8.6**.

   Analyze the bouncing ball (gravitational potential) problem using the WKB approximation.

   (a) Find the allowed energies, E_n, in therms of m, g, and h.

   (b) Compare the WKB approximation to the first four energies with the "exact" results that we found in class.

   (c) About how large would the quantum number n have to be to give the ball an average height of, say, 1 meter above the ground?

4. **Griffiths 8.8**. Consider a particle of mass m in the nth stationary state of the harmonic oscillator (angular frequency ω).

   (a) Find the turning point, x_2 (upward sloping turning point).

   (b) How far (d) could you go *above* the turning point before the error in the linearized potential, ($V(x_2) \approx E + V'(x_2)x$) reaches 1%? That is, if

$$\frac{V(x_2 + d) - V_{lin}(x_2 + d)}{V(x_2)} = 0.01,$$

   what is d?

(c) The asymptotic form of $Ai(z)$ is accurate to 1% as long as $z \leq 5$. For the d in part (b), determine the smallest n such that $\alpha d \geq 5$. (For any n larger than this there exists an overlap region in which the linearized potential is good to 1% *and* the large-z form of the Airy function is good to 1%.

5. **Griffiths 8.15**.

Consider the case of the symmetrical double well such as the one pictured in Figure 8.13. We are interested in bound states with $E < V(0)$.

(a) Write down the WKB wave functions in regions (i) $x < x_2$, (ii) $x_1 < x < x_2$, and (iii) $0 < x < x_1$. Impose the appropriate connection formulats at $x_1 and x_2$ (this has already been done, in Equation 8.46, for x_2; you will have to work out x_1 for yourself), to show that

$$\psi(x) \approx \begin{cases} \frac{D}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_2}^{x} |p(x')|dx'\right] & \text{(i)} \\ \frac{2D}{\sqrt{|p(x)|}} \sin\left[\frac{1}{\hbar}\int_{x}^{x_2} p(x')dx' + \frac{\pi}{4},\right], & \text{(ii)} \\ \frac{D}{\sqrt{|p(x)|}} \left[2\cos\theta e^{\frac{1}{\hbar}\int_{x}^{x_1} |p(x')|dx'} + \sin\theta e^{-\frac{1}{\hbar}\int_{x}^{x_1} |p(x')|dx'}\right], & \text{(iii)} \end{cases}$$

where

$$\theta \equiv \frac{1}{\hbar}\int_{x_1}^{x_2} p(x)dx. \tag{2}$$

(b) Because $V(x)$ is symmetric, we need only consider even (+) and odd (-) wave functions. In the former case $\psi'(0) = 0$, and in the latter case $\psi(0) = 0$. Show that this leads to the following quantization condition:

$$\tan\theta = \pm 2e^{\phi}, \tag{3}$$

where

$$\phi \equiv \frac{1}{\hbar}\int_{-x_1}^{x_1} |p(x')|dx'. \tag{4}$$

Equation 3 determines the (approximate) allowed energies (note that E comes into x_1 and x_2, so θ and ϕ are both functions of E).

(c) We are particularly interested in a high and/or broad central barrier, in which case ϕ is large, and e^{ϕ} is *huge*. Equation 3 then tells us that θ must be very close to a half-integer multiple of π. With this in mind, write $\theta = (n + 1/2)\pi + \epsilon$, where $|\epsilon| \ll 1$, and show that the quantization condition becomes

$$\theta \approx \left(n + \frac{1}{2}\right)\pi \mp \frac{1}{2}e^{-\phi}. \tag{5}$$

(d) Suppose each well is a parabola

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2(x+a)^2, & \text{if } x < 0, \\ \\ \frac{1}{2}m\omega^2(x-a)^2, & \text{if } x > 0. \end{cases} \tag{6}$$

Sketch this potential, find θ (Equation 2), and show that

$$E_n^{\pm} \approx \left(n + \frac{1}{2}\right)\hbar\omega \mp \frac{\hbar\omega}{2\pi}e^{-\phi} \tag{7}$$

Comment : If the central barrier were *impenetrable* ($\phi \to \infty$), we would simply have two detached harmonic oscillators, and the energies, $E_n = (n + \frac{1}{2})\hbar\omega$, would be doubly degenerate, since the particle could be in the left well or in the the right one. When the barrier becomes *finite* (putting the two wells into "communication:) the degeneracy is lifted. The even states (ψ_n^+) have slightly *lower* energy, and the odd ones (ψ_n^-) have slightly higher energy.

(e) Suppose the particle starts out in the *right* well - or, more precisely in a state of the form

$$\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_n^+ + \psi_n^-),$$

which, assuming the phases are picked in the "natural" way, will be concentrated in the right well. Show that it oscillates back and forth between the wells, with a period

$$\tau = \frac{2\pi^2}{\omega}e^{\phi}. \tag{8}$$

(f) Calculate ϕ, for the specific potential in part (d), and show that for $V(0) \gg E$, $\phi \sim m\omega a^2/\hbar$.