P443 HW #3
Due February 13, 2008

1. Griffiths 3.23. The Hamiltonian for a certain two level system is

$$\hat{H} = \epsilon(|\,1\rangle\langle 1\,| - |\,2\rangle\langle 2\,| + |\,1\rangle\langle 2\,| + |\,2\rangle\langle 1\,|),$$

where $|\,1\rangle, |\,2\rangle$ is an orthonormal basis and ϵ is a number with the dimension of energy. Find its eigenvalues and eigenvectors (as linear combinations of $|\,1\rangle$ and $|\,2\rangle$. What is the matrix $\mathbf{H}$ representing $\hat{H}$ with respect to this basis?

2. Griffiths 3.24. Let $\hat{Q}$ be an operator with a complete set of orthonormal eigenvectors:

$$\hat{Q}|\,e_n\rangle = q_n|\,e_n\rangle \quad (n = 1, 2, 3...).$$

Show that $\hat{Q}$ can be written in terms of its **spectral decomposition**:

$$\hat{Q} = \sum_n q_n|\,e_n\rangle\langle e_n\,|.$$

Hint : An operator is characterized by its action on all possible vectors, so what you must show is that

$$\hat{Q}|\,\alpha\rangle = \left\{\sum_n q_n|\,e_n\rangle\langle e_n\,|\right\}|\,\alpha\rangle,$$

for any vector $|\,\alpha\rangle$.

3. Griffiths 3.3. Show that if $\left\langle h\mid \hat{Q}h\right\rangle = \left\langle \hat{Q}h\mid h\right\rangle$ for all funcions h (in Hilbert space), then $\left\langle f\mid \hat{Q}g\right\rangle = \left\langle \hat{Q}f\mid g\right\rangle$ for all f and g (.e., the two definitions of "hermitian"

$$\begin{aligned}\left\langle f\mid \hat{Q}f\right\rangle &= \left\langle \hat{Q}f\mid f\right\rangle \\ \left\langle f\mid \hat{Q}g\right\rangle &= \left\langle \hat{Q}f\mid g\right\rangle\end{aligned}$$

are equivalent. *Hint* : First let $h = f + g$, and then let $h = f + ig$.

4. Griffiths 3.31. **Viral theorem**. Use the relationship

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle \hat{H}, \hat{Q}]\rangle + \langle\frac{\partial \hat{Q}}{\partial t}\rangle$$

to show that

$$\frac{d}{dt}\langle xp\rangle = 2\langle T\rangle - \langle x\frac{dV}{dx}\rangle,$$

where T is the kinetic energy ($H = T + V$). In a *stationary* state the left side is zero (why?) so

$$2\langle T\rangle = \langle x\frac{dV}{dx}\rangle.$$

This is called the **virial theorem**. Use it to prove that $\langle T\rangle = \langle V\rangle$ for stationary states of the harmonic oscillator.

5. Griffiths 3.33. Find the matrix elements $\langle n \mid x \mid n'\rangle$ and $\langle n \mid p \mid n'\rangle$ in the (orthonormal) basis of stationary states for the harmonic oscillator

$$\psi_n = \frac{1}{\sqrt{n!}}(a_+)^n\psi_0.$$

You already calculated the "diagonal" elements ($n = n'$) in Problem 2.12; use the same technique for the general case. Construct the corresponding (infinite) matrices, $\mathbf{X}$ and $\mathbf{P}$. Show that $(1/2m)\mathbf{P}^2 + (m\omega^2/2)\mathbf{X}^2 = \mathbf{H}$ is *diagonal*, in this basis. Are its diagonal elements what you would expect? *Partial answer* :

$$\langle n \mid x \mid n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{n'}\delta_{n,n'-1} + \sqrt{n}\delta_{n',n-1}).$$

6. Griffiths 3.38. The Hamiltonian for a certain three-level system is represented by the matrix

$$\mathbf{H} = \hbar\omega\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The other observables, A and B, are represented by the matrices

$$\mathbf{A} = \lambda\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad \mathbf{B} = \mu\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω, λ, and μ are positive real numbers.

(a) Find the eigenvalues and (normalized) eigenvectors of **H**, **A**, and **B**.

(b) Suppose the system starts out in the generic state

$$| S(0)\rangle = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

with $|c_1|^2 + |c_2|^2 + |c_3|^3 = 1$. Find the expectatio values (at $t = 0$) of H, A and B.

(c) What is $| S(t)\rangle$? If you measured the energy of this state (at time t), what values might you get, and what is the probability of each? Answer the same questions for A and B.

7. Charmonium

The ψ meson is a bound state of a charmed quark (c) and its antiparticle ($\bar{c}$). A linear potential is sometimes used to represent phenomonologically the force binding them together. Then the radial Schrodinger equation for the system is

$$\frac{-\hbar^2}{2m}\frac{\partial^2 u}{\partial r^2} + \alpha|r|u = Eu$$

Here m is the reduced mass of the system.

(a) Find the energy of the ground state and the first radially excited state in terms of $\hbar, \alpha$, and m. Assume that the system is nonrelativistic.

The measured mass of the ground state $\psi(1s)$ is $m_g = 3.1GeV/c^2$ and the mass of the first radially excited state $\psi_(2s)$ is $m_1 = 3.685GeV/c^2$.

(b) Determine the reduced mass m, (and also the mass of the charmed quark), and α so that the energies you calculate are consistent with the measured masses.

(c) What is the characteristic size of the bound state?

8. Rotations

Consider a spin 1/2 system with state vector

$$| \psi\rangle = a| +\frac{1}{2}\rangle + b| -\frac{1}{2}\rangle$$

The angular momentum operator $\hat{L}_y$ with expectation value corresponding to the y-component of angular momentum is also the generator of rotations about the y-axis. The matrix representation of L_y is

$$\begin{pmatrix} \left\langle +\frac{1}{2} \mid L_y \mid +\frac{1}{2} \right\rangle & \left\langle +\frac{1}{2} \mid L_y \mid -\frac{1}{2} \right\rangle \\ \left\langle -\frac{1}{2} \mid L_y \mid +\frac{1}{2} \right\rangle & \left\langle -\frac{1}{2} \mid L_y \mid -\frac{1}{2} \right\rangle \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

And the operator

$$R(\theta) = e^{iL_y\theta/\hbar}$$

is the rotation matrix.

(a) Construct this 2×2 matrix representing rotation by θ about the y-axis.

(b) Is $\hat{L}_y$ hermitian?

(c) Is $R(\theta)$ unitary?