Physics 443, Solutions to PS 2

1. Griffiths 2.12.

The raising and lowering operators are

$$a_\pm = \frac{1}{\sqrt{2m\omega\hbar}} \left(\mp i\hat{p} + m\omega\hat{x}\right)$$

where $\hat{p}$ and $\hat{x}$ are momentum and position operators. Then

$$\begin{aligned}
\hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
\hat{p} &= i\sqrt{\frac{m\omega\hbar}{2}}(a_+ - a_-)
\end{aligned}$$

The expectation value of the position operator is

$$\begin{aligned}
\langle x \rangle &= \langle \psi_n \mid \hat{x} \mid \psi_n \rangle \\
&= \left\langle \psi_n \mid \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \mid \psi_n \right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\langle \psi_n \mid (a_+) \mid \psi_n \rangle + \langle \psi_n \mid (a_-) \mid \psi_n \rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\langle \psi_n \mid \psi_{n+1} \rangle + \sqrt{n-1}\langle \psi_n \mid \psi_{n-1} \rangle\right) \\
&= 0
\end{aligned}$$

Similarly, the expectation value of the momentum operator is

$$\begin{aligned}
\langle p \rangle &= \langle \psi_n \mid \hat{p} \mid \psi_n \rangle \\
&= \left\langle \psi_n \mid i\sqrt{\frac{m\omega\hbar}{2}}(a_+ - a_-) \mid \psi_n \right\rangle \\
&= 0
\end{aligned}$$

Expectation values of x^2 and p^2 are not zero.

$$\begin{aligned}
\langle x^2 \rangle &= \left\langle \psi_n \mid \hat{x}^2 \mid \psi_n \right\rangle \\
&= \left\langle \psi_n \mid \frac{\hbar}{2m\omega}(a_+^2 + a_-^2 + a_+a_- + a_-a_+) \mid \psi_n \right\rangle \\
&= \frac{\hbar}{2m\omega} \langle \psi_n \mid (a_+a_- + a_-a_+) \mid \psi_n \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\hbar}{2m\omega}\left\langle \psi_n \mid \frac{2}{\hbar\omega} H \mid \psi_n \right\rangle \\
&= \frac{\hbar}{2m\omega}\left\langle \psi_n \mid \frac{2}{\hbar\omega}(n+\frac{1}{2})\hbar\omega \mid \psi_n \right\rangle \\
&= \frac{\hbar}{2m\omega}(2n+1)
\end{aligned}
$$

where we have used the relationship between Hamiltonian operator and $a_{\pm}$, namely

$$
\begin{aligned}
H &= \hbar\omega(a_+a_- + \frac{1}{2}) \\
H &= \hbar\omega(a_-a_+ - \frac{1}{2})
\end{aligned}
$$

Then

$$
\begin{aligned}
\langle p^2 \rangle &= \left\langle \psi_n \mid \hat{p}^2 \mid \psi_n \right\rangle \\
&= \left\langle \psi_n \mid \frac{m\omega\hbar}{2}(-a_+^2 - a_-^2 + a_+a_- + a_-a_+) \mid \psi_n \right\rangle \\
&= \frac{m\omega\hbar}{2}\langle \psi_n \mid (a_+a_- + a_-a_+) \mid \psi_n \rangle \\
&= \frac{m\omega\hbar}{2}\left\langle \psi_n \mid \frac{2}{\hbar\omega} H \mid \psi_n \right\rangle \\
&= \frac{m\omega\hbar}{2}(2n+1)\langle \psi_n \mid \psi_n \rangle \\
&= \frac{m\omega\hbar}{2}(2n+1)
\end{aligned}
$$

We have that

$$\sigma_x = \sqrt{\langle (x - \langle x \rangle) \rangle} = \sqrt{\langle x^2 \rangle} = \sqrt{\frac{\hbar}{2m\omega}(2n+1)}$$

$$\sigma_p = \sqrt{\langle (p - \langle p \rangle) \rangle} = \sqrt{\langle p^2 \rangle} = \sqrt{\frac{m\omega\hbar}{2}(2n+1)}$$

and then

$$\sigma_x \sigma_p = (2n+1)\frac{\hbar}{2}$$

The kinetic energy

$$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{\hbar\omega}{4}(2n+1)$$

2. Griffiths 2.14.

The wave function is initially in the ground state of the oscillator with classical frequency ω. The wave function is

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

The spring constant changes instantaneously but the wav function does not. So immediately after the change in spring constant the wave function remains the same. But it is no longer an eigenfunction of the hamiltonian operator. However, any function can be expressed as a linear combination of the new eigenfunctions, ψ'_n and we can write that

$$\psi_0(x) = \sum_{n=0}^{n=\infty} a_n \psi'_n(x)$$

The probability that we will find the oscillator in the n^{th} state, with energy E'_n is $|a_n|^2$. After the change, the minimum energy state is $E'_0 = \frac{1}{2}\hbar\omega' = \hbar\omega$, (since $\omega' = 2\omega$) so the probablity that a measurement of the energy would still return the value $\hbar\omega/2$ is zero. Since the eigenfunctions are orthonormal ($\int \psi'_n \psi'_m dx = \delta_{nm}$) we can determine the coefficients

$$\begin{aligned}
a_n &= \int_{-\infty}^{\infty} \psi'_n(x)\psi_0(x)dx \\
a_0 &= \int_{-\infty}^{\infty} \left(\frac{m\omega'}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega'}{2\hbar}x^2} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2} dx \\
&= \left(\frac{m\omega'}{\pi\hbar}\right)^{\frac{1}{4}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-(\frac{m\omega'}{2\hbar}+\frac{m\omega}{2\hbar})x^2} dx \\
&= \left(\frac{\sqrt{2}m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-(\frac{3m\omega}{2\hbar})x^2} dx \\
&= \left(\frac{\sqrt{2}m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \left(\frac{2\pi\hbar}{3m\omega}\right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{2\sqrt{2}}{3}\right)^{\frac{1}{2}} \\
&= 0.971
\end{aligned}
$$

The probability of measuring energy $E_0' = \hbar\omega'/2 = \hbar\omega$ is the probability that the oscillator is in the state ψ_0'. The probability that the oscillator is in the ground state is

$$|a_0|^2 = 0.943$$

3. Griffiths 2.26.

Using the definition of the Fourier transform, we have

$$
\begin{aligned}
\text{Answer} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ \delta(x) \exp(-ikx), \\
&= \frac{1}{\sqrt{2\pi}} e^{-ikx}|_{x=0}, \\
&= \frac{1}{\sqrt{2\pi}}.
\end{aligned}
$$

We therefore have that

$$
\begin{aligned}
\delta(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(ikx)\ dk, \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ e^{ikx}. \qquad (1)
\end{aligned}
$$

4. Griffiths 2.38.

(a) We write the wave function ψ_1, which is the ground state wave function for the well of width a as a linear combination of the eigenfunction, (ψ_n^prime) of the well of width $2a$.

$$\psi_1 = \sum_{n=1}^{\infty} a_n \psi_n'$$

Solve for the coefficients

$$
\begin{aligned}
a_n &= \int_0^{2a} (\psi_n')^* \psi_1 dx \\
&= \int_0^a \sqrt{\frac{2}{2a}} \sin \frac{n\pi x}{2a} \sqrt{\frac{2}{a}} \sin \frac{\pi x}{a} dx
\end{aligned}
$$

We only integrate from 0 to a because $\psi_1(x)$ is zero for $x < 0$ and $x > a$. Then

$$\begin{aligned} a_n &= \frac{\sqrt{2}}{a}\int_0^{\pi/2}(\sin ny)(\sin 2y)\left(\frac{2a}{\pi}\right)dy \\ &= \frac{2\sqrt{2}}{\pi}\left[\frac{\sin(2-n)y}{2(2-n)} - \frac{\sin(2+n)y}{2(2+n)}\right]_0^{\pi/2} \qquad (n \neq 2) \end{aligned}$$

If $n = 2$, the $\int_0^{\pi/2}\sin^2 2ydy = \frac{\pi}{4}$ and $a_2 = \frac{\sqrt{2}}{2}$, and the probability to measure $E_2' = \frac{\hbar^2}{2m}\left(\frac{\pi}{2a}\right)^2$, is $|a_2|^2 = \frac{1}{2}$. Since the sum of probabilities for any energy is 1, no other can be more probable than E_2'.

(b) For all other even n, $a_n = 0$. For n odd

$$\begin{aligned} a_n &= \frac{2\sqrt{2}}{2\pi}\left[\frac{\sin(n\pi/2)}{(2-n)} + \frac{\sin(n\pi/2)}{(2+n)}\right] \\ &= \frac{4\sqrt{2}}{\pi}\frac{\sin(n\pi/2)}{(4-n^2)} \end{aligned}$$

Then

$$|a_n|^2 = \frac{32}{\pi^2(4-n^2)^2}$$

and $|a_1|^2 = 0.36$ The next most probable result is E_1' with probability 0.36.

(c)

$$\begin{aligned} \langle H \rangle &= \int_0^a \psi_1^* H \psi_1 dx \\ &= \frac{\hbar^2\pi^2}{2ma^2} \end{aligned}$$

7. Time dependence.

   Show that if $\hat{Q}$ is an operator that does not involve time explicitly, and if ψ is any eigenfunction of $\hat{H}$, that the expectation value of $\hat{Q}$ in the state of ψ is independent of time.

[We start with the Griffith's Equation 3.148

$$\frac{d}{dt}\langle Q\rangle = \frac{i}{\hbar}\langle[H,Q]\rangle + \langle\frac{dQ}{dt}\rangle,$$

Where the last term is zero because Q has no explicit time dependence,

$$\begin{aligned}\frac{d}{dt}\langle Q\rangle &= \frac{i}{\hbar}\langle\psi|HQ - QH|\psi\rangle,\\ &= \epsilon_\psi\langle\psi|(Q-Q)\psi\rangle \quad \text{using } \psi \text{ is stationary},\\ &= 0 \end{aligned} \tag{2}$$

8. Collapse of the wave function.

   Consider a particle in the infinite square well potential from problem 4.

   (a) Show that the stationary states are $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$ and the energy spectrum is $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$ where the width of the box is a.
   [The time independent Schrodinger's equation for a particle in an infinite square well is

   $$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

   Substitution of the proposed solution $\psi_n(x)$ gives

   $$\begin{aligned}-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) &= E\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\\ \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) &= E\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)\\ \rightarrow E_n &= \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2]\end{aligned}$$

   (b) Suppose we now make a measurement that locates a particle initially in state $\psi_n(x)$ so that it is now in the position $a/2 - \epsilon/2 \leq x \leq a/2 + \epsilon/2$ and described by the state α. In the limit where $\epsilon \ll a$, the result of the measurement projects the system onto a superposition of eigenstates of energy. The probability of finding the particle in any eigenstate is $P(E_n) = |\langle\psi_n \mid \alpha\rangle|^2$. A reasonable estimate of the state $\mid \alpha\rangle$ is $\psi_\alpha(x) = \sqrt{\epsilon}\delta^\epsilon(x - a/2)$ where $\delta^{(\epsilon)}(x-a/2) = 1/\epsilon$ for $a/2-\epsilon/2 \leq x \leq a/2+\epsilon/2$ and $\delta^\epsilon(x-\frac{a}{2}) = 0$ everywhere else. Calculate the probability $P(E_n)$.

[Any solution to (the time dependent) Schrodinger's equation can be written as a linear combination of energy eigenstates. (The energy eigenstates form a complete set.) So we can write

$$
\begin{aligned}
\psi_\alpha(x) &= \sum_n^\infty c_n\psi_n \\
\rightarrow c_n &= \int_{-\infty}^{\infty} \psi_\alpha(x)\psi_n(x)^* dx \\
&= \int_{a/2-\epsilon/2}^{a/2+\epsilon/2} \left(\frac{1}{\sqrt{\epsilon}}\right) \psi_n(x)^* dx \\
&= -\sqrt{\frac{2}{\epsilon a}}\frac{a}{n\pi}\left[\cos\frac{n\pi x}{a}\right]_{a/2-\epsilon/2}^{a/2+\epsilon/2} \\
&= -\sqrt{\frac{2}{\epsilon a}}\frac{a}{n\pi}\left[\cos(\frac{n\pi}{2}+\frac{n\pi\epsilon}{2a}) - \cos(\frac{n\pi}{2}-\frac{n\pi\epsilon}{2a})\right] \\
&= 2\sqrt{\frac{2}{\epsilon a}}\frac{a}{n\pi}\left[\sin(\frac{n\pi}{2})\sin(\frac{n\pi\epsilon}{2a})\right] \\
&= \begin{cases} (-1)^{(n-1)/2}(\frac{2}{n\pi})\sqrt{\frac{2a}{\epsilon}}\sin(\frac{n\pi\epsilon}{2a}), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}
\end{aligned}
$$

The probability of finding the particle in the n^{th} energy eigenstate is

$$
P(E_n) = |c_n|^2 = \begin{cases} (\frac{2}{n\pi})^2\frac{2a}{\epsilon}\sin^2(\frac{n\pi\epsilon}{2a}), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \tag{3}
$$

As ϵ shrinks, and the particle is more localized, the probability that it will be found in a higher energy state increases. In the limit $\epsilon \ll a$,

$$
P(E_n) \rightarrow \begin{cases} \frac{2\epsilon}{a}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \tag{4}
$$

the particle is equally likely to be found in all n odd states.]