Physics 443, Solutions to PS 12

1 Griffiths 10.1

(a)

We need to show that

$$\Phi_n(s,t) = \sqrt{\frac{2}{w}} \sin\left(\frac{n\pi}{w}x\right) e^{i(mvx^2 - 2E_n^i at)/2\hbar w}$$

satisfies Schrodinger's equation for the infinite square well

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Phi_n = i\hbar\frac{\partial}{\partial t}\Phi_n$$

It is clear that $\Phi(x,t)$ satsifies the boundary conditions, namely $\Phi(0,t) = \Phi(w,t) = 0$ for all t.

$$\frac{\partial}{\partial x}\Phi_n = \sqrt{\frac{2}{w}}\left(\frac{n\pi}{w}\cos\left(\frac{n\pi}{w}x\right) + \frac{imvx}{\hbar w}\sin\left(\frac{n\pi}{w}x\right)\right) e^{i(mvx^2-2E_n^i at)/2\hbar w}$$

and

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Phi_n &= -\frac{\hbar^2}{2m}\sqrt{\frac{2}{w}}\left(-\left(\frac{n\pi}{w}\right)^2\sin\left(\frac{n\pi}{w}x\right) + 2\frac{n\pi}{w}\frac{imvx}{\hbar w}\cos\left(\frac{n\pi}{w}x\right) + \frac{imv}{\hbar w}\sin\left(\frac{n\pi}{w}x\right)\right. \\ &\quad \left. -\left(\frac{mvx}{\hbar w}\right)^2\sin\left(\frac{n\pi}{w}x\right)\right) e^{i(mvx^2-2E_n^i at)/2\hbar w}\end{aligned}$$

Then

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\Phi_n &= i\hbar\left(-\frac{1}{2}\sqrt{\frac{2}{w^3}}v\sin\frac{n\pi}{w}x - \sqrt{\frac{2}{w}}\frac{n\pi x}{w^2}v\cos\frac{n\pi}{w}x\right. \\ &\quad \left. -\sqrt{\frac{2}{w}}\sin\frac{n\pi}{w}x\left(\frac{iE_n^i a}{\hbar w} + \frac{i(mvx^2 - 2E_n^i at)}{2\hbar w^2}v\right)\right) e^{i(mvx^2-2E_n^i at)/2\hbar w}\end{aligned}$$

We need to establish that the last two equations are equal to each other. Dividing out the exponential factor on both sides and setting imaginary parts equal we get

$$\begin{aligned}-\frac{\hbar^2}{2m}\sqrt{\frac{2}{w}}\left(\frac{2n\pi mvx}{\hbar w^2}\cos\frac{n\pi}{w}x + \frac{mv}{\hbar w}\sin\frac{n\pi}{w}x\right) &= -\hbar\left(\frac{1}{2}\sqrt{\frac{2}{w^3}}v\sin\frac{n\pi}{w}x - \sqrt{\frac{2}{w}}\frac{n\pi xv}{w^2}\cos\frac{n\pi}{w}x\right) \\ -\hbar\sqrt{\frac{2}{w}}\left(\frac{n\pi vx}{w^2}\cos\frac{n\pi}{w}x + \frac{v}{2w}\sin\frac{n\pi}{w}x\right) &= -\hbar\left(\frac{1}{2}\sqrt{\frac{2}{w^3}}v\sin\frac{n\pi}{w}x - \sqrt{\frac{2}{w}}\frac{n\pi xv}{w^2}\cos\frac{n\pi}{w}x\right)\end{aligned}$$

We see that coefficients of sin and cos terms are equal. Next we equate real parts of $-\hbar^2/2m\frac{\partial^2\Phi}{\partial x^2}$ and $i\hbar\frac{\partial\Phi}{\partial t}$.

$$\begin{aligned}-\frac{\hbar^2}{2m} \quad & \sqrt{\frac{2}{w}}\left(-\left(\frac{n\pi}{w}\right)^2 \sin\left(\frac{n\pi}{w}x\right) - \left(\frac{mvx}{\hbar w}\right)^2 \sin\left(\frac{n\pi}{w}x\right)\right) \\ & = -i\hbar\sqrt{\frac{2}{w}} \sin\frac{n\pi}{w}x\left(\frac{iE_n^i a}{\hbar w} + \frac{i(mvx^2 - 2E_n^i at}{2\hbar w^2}v\right)\end{aligned}$$

Rearrange left and right sides and divide out $\sin\frac{n\pi}{w}x$ we get

$$\frac{\hbar^2}{2m}\left(\left(\frac{n\pi}{w}\right)^2 + \left(\frac{mvx}{\hbar w}\right)^2\right) = \left(\frac{E_n^i a}{w} + \frac{(mvx^2 - 2E_n^i at}{2w^2}v\right)$$

The term on the left that multiplies x^2 is the same as the term on the right that multiplies x^2. That leaves

$$\begin{aligned}\frac{\hbar^2}{2m}\left(\frac{n\pi}{w}\right)^2 &= \left(\frac{E_n^i a}{w} - \frac{E_n^i at}{w^2}v\right) \\ &= \frac{E_n^i a(w - tv)}{w^2} \\ &= \frac{\hbar^2 n^2\pi^2}{2ma^2}\frac{a(w - tv)}{w^2}\end{aligned}$$

And since $w - tv = a$, left equals right.

(b)

The c_n are independent of time so

$$\Psi(x, 0) = \sum_{n=1}^{\infty} c_n \Phi_n(x, 0)$$

$$\sqrt{\frac{2}{a}} \sin\frac{\pi}{a}x = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin\frac{n\pi}{a}xe^{imvx^2/2\hbar a}$$

Multiply both sides $\exp(-imvx^2/2\hbar a)\sin\frac{m\pi}{a}x$ and integrate from 0 to a.

$$\int_0^a e^{-imvx^2/2\hbar a} \sin\frac{\pi}{a}x \sin\frac{m\pi}{a}xdx = \sum_{n=1}^{\infty}\int_0^a c_n \sin^2\frac{m\pi}{a}xdx = \frac{a}{2}c_m$$

Let $z = \pi x/a$ and $\alpha = mva/2\pi^2\hbar$ and we get that

$$c_m = \frac{2}{\pi}\int_0^\pi e^{-i\alpha z^2}\sin z\sin(mz)dz$$

(c)

If the well expands to twice its original width and $w(T_e) = 2a$, then $T_e = (2a - a)/v = a/v$. The internal time is $T_i = 2\pi/\omega$ and $\omega = (E_n/\hbar)$. Then

$$T_i = \frac{2\pi\hbar}{\pi^2\hbar^2/2ma^2} = \frac{4ma^2}{\hbar\pi}$$

and

$$\frac{T_i}{T_e} = \frac{4mav}{\hbar\pi} = 8\pi\alpha$$

In the adiabatic limit $T_i \ll T_e$ which corresponds to α
1 and $\exp(-i\alpha z^2) \sim 1$. Then

$$c_m = \frac{2}{\pi}\int_0^\pi \sin z\sin(mz)dz = \delta_{m1}$$

That is, $c_1 = 1$ and $c_n = 0$ for $c \neq 1$, and

$$\Psi(x,t) = \Phi_1(x,t)$$

In the adiabatic limit $\Psi(x,t)$ is always in the $n = 1$ state with wave function and energy corresponding to the width of the well at t.

(d)

The phase factor

$$\begin{aligned}
\theta(t) &= -\frac{1}{\hbar}\int_0^t E(t')dt' \\
&= -\frac{1}{\hbar}\int_0^t \frac{\hbar^2\pi^2}{2m(a+vt')^2}dt' \\
&= -\frac{\hbar\pi^2}{2m}\int_0^t \frac{1}{(a+vt')^2}dt' \\
&= \frac{\hbar\pi^2}{2m}\frac{1}{v(a+vt')}|_0^t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\hbar\pi^2}{2m}\left(\frac{1}{v(a+vt)} - \frac{1}{va}\right) \\
&= \frac{\hbar\pi^2}{2m}\frac{1}{v}\left(\frac{1}{w} - \frac{1}{a}\right) \\
&= \frac{\hbar\pi^2}{2m}\frac{(a-w)}{wav} \\
&= \frac{\hbar\pi^2}{2m}\frac{t}{wa} \\
&= \frac{\hbar^2\pi^2}{2ma^2}\frac{at}{\hbar w} \\
&= E_1^i at/\hbar\omega
\end{aligned}
$$

2 Griffiths 10.3

1. The geometric phase is

$$
\begin{aligned}
\gamma_n(t) &= i\int_{R_i}^{R_f}\left\langle \psi_n \mid \frac{\partial\psi_n}{\partial R}\right\rangle dR \\
&= i\int_{w_1}^{w_2} dw \int_0^w dx \sqrt{\frac{2}{w}}\sin\left(\frac{n\pi}{w}x\right)\frac{\partial}{\partial w}\left(\sqrt{\frac{2}{w}}\sin\left(\frac{n\pi}{w}x\right)\right) \\
&= i\int_{w_1}^{w_2} dw \int_0^w dx \sqrt{\frac{2}{w}}\sin\left(\frac{n\pi}{w}x\right)(-\sqrt{\frac{2}{w^3}}\left(\frac{1}{2}\sin\left(\frac{n\pi}{w}x\right) + \frac{n\pi x}{w}\cos\left(\frac{n\pi}{w}x\right)\right) \\
&= -i\int_{w_1}^{w_2} dw \int_0^w dx \frac{2}{w^2}\left(\frac{1}{2}\sin^2\left(\frac{n\pi}{w}x\right) + \frac{n\pi x}{w}\sin\left(\frac{n\pi}{w}x\right)\cos\left(\frac{n\pi}{w}x\right)\right) \\
&= -i\int_{w_1}^{w_2} dw \left[\frac{1}{2w} + \int_0^w dx \frac{2n\pi x}{w^3}\sin\left(\frac{n\pi}{w}x\right)\cos\left(\frac{n\pi}{w}x\right)\right] \\
&= -i\int_{w_1}^{w_2} dw \left[\frac{1}{2w} - \frac{1}{2w}\right] \\
&= 0
\end{aligned}
$$

2. The dynamic phase change

$$
\begin{aligned}
\theta &= \frac{1}{\hbar}\int E_n(t')dt' \\
&= \frac{1}{\hbar}\int \frac{n^2\pi^2\hbar^2}{2mw(t')^2}dt'
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\hbar}\frac{n^2\pi^2\hbar^2}{2m}\int\frac{1}{(a+vt')^2}dt' \\
&= \frac{n^2\pi^2\hbar}{2m}\left(\frac{-1}{v(a+vt')}\Big|_0^t\right) \\
&= \frac{n^2\pi^2\hbar}{2m}\left(\frac{-1}{v(a+vt)}+\frac{1}{va}\right) \\
&= -\frac{n^2\pi^2\hbar}{2mv}\left(\frac{1}{w_2}-\frac{1}{w_1}\right) \\
&= -\frac{n^2\pi^2\hbar}{2mv}\left(\frac{w_1-w_2}{w_1w_2}\right) \\
&= \frac{n^2\pi^2\hbar}{2m}\left(\frac{t}{w_1w_2}\right)
\end{aligned}
$$

The dynamic phase is evidently $1/\hbar$ times the geometric mean of the energies at starting and ending widths of the well times the time.

$$\theta(t) = \frac{1}{\hbar}\sqrt{E_n(0)E_n(t)}t$$

3. The geometric phase is zero for any expansion or contraction so therefore so is Berry's phase for any cycle.

3 Griffiths 10.6

We have that

$$\mathbf{B}(t) = B_0[\sin\alpha\cos(\omega t)\hat{\imath} + \sin\alpha\sin(\omega t)\hat{\jmath} + \cos\alpha\hat{k}]$$

The hamiltonian is

$$H(t) = \frac{e}{m}\mathbf{B}\cdot\mathbf{S}$$

and for spin 1 (see problem set #4, problem 2)

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix},\quad S_y = \frac{i\hbar}{\sqrt{2}}\begin{pmatrix}0&-1&0\\1&0&-1\\0&1&0\end{pmatrix},\quad S_z = \hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

Then

$$H(t) = \frac{eB_0\hbar}{\sqrt{2}m}\begin{pmatrix} \sqrt{2}\cos\alpha & \sin\alpha e^{-i\omega t} & 0 \\ \sin\alpha e^{i\omega t} & 0 & \sin\alpha e^{-i\omega t} \\ 0 & \sin\alpha e^{i\omega t} & -\sqrt{2}\cos\alpha \end{pmatrix}$$

We need to find the eigenvector with eigenvalue $+\hbar\omega_1$ where $\omega_1 = \frac{eB_0}{m}$. Let

$$\chi_+ = c\begin{pmatrix} a \\ 1 \\ b \end{pmatrix}$$

. The eigenvalue equation is

$$\begin{aligned} H\chi_+ &= \hbar\chi_+ \\ &= \frac{\omega_1\hbar}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\cos\theta & \sin\theta e^{-i\phi} & 0 \\ \sin\theta e^{i\phi} & 0 & \sin\theta e^{-i\phi} \\ 0 & \sin\theta e^{i\phi} & -\sqrt{2}\cos\theta \end{pmatrix} c\begin{pmatrix} a \\ 1 \\ b \end{pmatrix} = \hbar\omega_1 c\begin{pmatrix} a \\ 1 \\ b \end{pmatrix} \end{aligned}$$

where $\theta = \alpha$, and $\phi = \omega t$. We get

$$\begin{aligned} a\sqrt{2}\cos\theta + \sin\theta e^{-i\phi} &= a\sqrt{2} \\ \sin\theta e^{i\phi} - b\sqrt{2}\cos\theta &= b\sqrt{2} \end{aligned}$$

from which we conclude that

$$\begin{aligned} a &= \frac{\sin\theta e^{-i\phi}}{\sqrt{2}(1-\cos\theta)} = \frac{1}{\sqrt{2}}\cot\frac{\theta}{2}e^{-i\phi} \\ b &= \frac{\sin\theta e^{i\phi}}{\sqrt{2}(1+\cos\theta)} = \frac{1}{\sqrt{2}}\tan\frac{\theta}{2}e^{i\phi} \end{aligned}$$

The constant c is chosen so that $\chi_+^\dagger\chi_+ = 1$. That is

$$\begin{aligned} |c|^2(|a|^2+1+|b|^2) &= 1 \\ &= |c|^2\frac{1}{2}(\cot^2\frac{\theta}{2}+2+\tan^2\frac{\theta}{2}) \\ &= |c|^2\frac{1}{2}\frac{(\cos^4\frac{\theta}{2}+2\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}+\sin^4\frac{\theta}{2})}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} \\ &= |c|^2\frac{1}{2}\frac{\left((\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2})^2\right)}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} \\ &= \frac{|c|^2}{\frac{1}{2}\sin^2\theta} \end{aligned}$$

So $|c| = \frac{\sin\theta}{\sqrt{2}}$ and

$$\chi_+ = \begin{pmatrix} \cos^2\frac{\theta}{2}e^{-i\phi} \\ \\ \frac{\sin\theta}{\sqrt{2}} \\ \\ \sin^2\frac{\theta}{2}e^{i\phi} \end{pmatrix}$$

To calculate the phase

$$\begin{aligned} \nabla\chi_+ &= \frac{\partial\chi_+}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\chi_+}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\chi_+}{\partial\phi}\hat{\phi} \\ &= \frac{1}{r}\begin{pmatrix} -\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi} \\ \\ \frac{\cos\theta}{\sqrt{2}} \\ \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}e^{i\phi} \end{pmatrix}\hat{\theta} + \frac{-i}{r\sin\theta}\begin{pmatrix} \cos^2\frac{\theta}{2}e^{-i\phi} \\ \\ 0 \\ \\ -\sin^2\frac{\theta}{2}e^{i\phi} \end{pmatrix}\hat{\phi} \end{aligned}$$

Then

$$\begin{aligned} \langle\chi_+ \mid \nabla\chi_+\rangle &= \frac{1}{r}\left((-\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2})\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right)\hat{\theta} + \frac{-i}{r}\frac{(\cos^4\frac{\theta}{2} - \sin^4\frac{\theta}{2})}{\sin\theta}\hat{\phi} \\ &= \frac{-i}{r}\frac{\cos\theta}{\sin\theta}\hat{\phi} \end{aligned}$$

Next

$$\nabla\times\langle\chi_+ \mid \nabla\chi_+\rangle = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{-i}{r}\cot\theta\right)\right]\hat{r} = \frac{i}{r^2}\hat{r}$$

Finally

$$\gamma_+(T) = i\int\frac{i}{r^2}\hat{r}\cdot d\mathbf{a} = -\Omega$$

where Ω is the solid angle on the sphere swept out by $\mathbf{B}$ in one cycle.