Physics 443, Solutions to PS 11

1.**Griffiths 9.1**

Hydrogen atom is placed in a time dependent electric field $E = E(t)\hat{k}$. The perturbation is given by $H = eE(t)z$.

$$\begin{aligned}
\psi_{100} &= \sqrt{\frac{1}{\pi a^3}} \exp\left(\frac{-r}{a}\right), \\
\psi_{200} &= \sqrt{\frac{1}{2\pi a}\frac{1}{2a}} \left(1 - \frac{r}{2a}\right) \exp\left(\frac{-r}{2a}\right), \\
\psi_{211} &= -\sqrt{\frac{1}{\pi a}\frac{1}{8a^2}} r \exp\left(\frac{-r}{2a}\right) \sin\theta e^{i\phi}, \\
\psi_{210} &= \sqrt{\frac{1}{2\pi a}\frac{1}{4a^2}} r \exp\left(\frac{-r}{2a}\right) \cos\theta, \\
\psi_{21-1} &= \sqrt{\frac{1}{\pi a}\frac{1}{8a^2}} r \exp\left(\frac{-r}{2a}\right) \sin\theta e^{-i\phi}.
\end{aligned}$$

By inspection or calculation, you will find that all the matrix elements are zero except

$$\langle 100|H|210\rangle = -\left(\frac{2^8}{\sqrt{2}3^5}\right) eEa.$$

2. **Griffiths 9.11**

We shall find the decay rates from the A coefficients which are

$$A = \frac{\omega^3}{3\pi\epsilon_0\hbar c^3}|\mathcal{P}|^2.$$

Refer to the previous problem for explicit representation of the hydrogenic wavefunctions. Using that $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$, you can convince yourself that all the matrix elements between ψ_{100} and ψ_{200} are zero. Therefore the state ψ_{200} has an infinite lifetime. For the state ψ_{210} we have that the x and y matrix elements are zero (think selection rules!) while (see previous problem)

$$\langle 100|z|210\rangle = \frac{2^8 a}{\sqrt{2}3^5},$$

from which we can calculate the A coefficient as

$$A = \frac{\omega^3}{3\pi\epsilon_0\hbar c^3}\frac{2^{15}}{3^{10}}(qa)^2.$$

We now consider the case of ψ_{211} and ψ_{21-1}, where the z matrix element is zero, and up to a phase, the x and y components are equal. We have that $|\mathcal{P}|^2 = 2(qa)^2 2^{14} 3^{-10}$. Notice that this is the same as what we calculated for the ψ_{210} state. Using that $\hbar\omega = E_2 - E_1 = 0.75E_1$, we have that $\tau = 1/A = 1.6 \text{ x } 10^{-9}$ sec.

3.**Griffiths 9.14**

The allowed decay routes are

$$|300\rangle \longrightarrow |210\rangle \longrightarrow |100\rangle$$
$$|300\rangle \longrightarrow |211\rangle \longrightarrow |100\rangle$$
$$|300\rangle \longrightarrow |21-1\rangle \longrightarrow |100\rangle$$

Recall the selection rules that $\Delta m = \pm 1, 0$ and $\Delta l = \pm 1$. Also note that on pg. 360 Griffiths derives the m-selection rules. We have if $\Delta m = \pm 1$ then the z matrix element is zero, while the x and y terms are equal up to a phase. While if $\Delta m = 0$ then the x and y matrix elements are zero. Using these rules, notice that $|\langle 210|\mathbf{r}|300\rangle|^2 = |\langle 210|z|300\rangle|^2$, and $|\langle 21 \pm 1|\mathbf{r}|300\rangle|^2 = 2|\langle 21 \pm 1|x|300\rangle|^2$. Comparing the integrals, you can notice that the ratio between $\langle 210|z|300\rangle$ and $\langle 21 \pm 1|x|300\rangle$ is $\sqrt{2}$. We therefore have that $|\langle 210|\mathbf{r}|300\rangle|^2 = |\langle 21 \pm 1|\mathbf{r}|300\rangle|^2$. This means that all three transition rates are equal, and that each route has an equal probability of 1/3. To calculate the lifetime, we add the A coefficients to get that

$$A = 3\left(\frac{\omega^3 e^2 |\langle \mathbf{r}\rangle|^2}{3\pi\epsilon_0\hbar c^3}\right)$$
$$|\langle \mathbf{r}\rangle|^2 = \frac{2^{15}3^7}{5^{12}}a^2$$
$$\hbar\omega = E_3 - E_2 = -\frac{5}{36}E_1.$$
$$\tau = \frac{1}{A} = 1.58 \text{ x } 10^{-7} \text{ sec.}$$

4. **Griffiths 9.20**

(a) Manipulations with the Pauli matrices give that

$$\mathcal{H} = -\vec{\mu}\cdot\vec{B} = -\frac{\gamma\hbar}{2}\begin{pmatrix} B_0 & B_{rf}e^{i\omega t} \\ B_{rf}e^{-i\omega t} & -B_0 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega e^{i\omega t} \\ \Omega e^{-i\omega t} & -\omega_0 \end{pmatrix}.$$

(b) Using the time dependent Schrodinger equation for a spinor $\chi(t) = (a(t), b(t))$, we can write:

$$-\frac{\hbar}{2}\begin{pmatrix} \omega_0 & \Omega e^{i\omega t} \\ \Omega e^{-i\omega t} & -\omega_0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = i\hbar\frac{\partial}{\partial t}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix} = \frac{i}{2}\begin{pmatrix} a(t)\omega_0 + b(t)\Omega e^{i\omega t} \\ a(t)\Omega e^{-i\omega t} - b(t)\omega_0 \end{pmatrix}.$$

(c) Solving these coupled differential equations is done by first differentiating the $\dot{b}(t)$ equation,

$$\ddot{b} = \frac{i}{2}\left[(\dot{a} - i\omega a)\Omega e^{-i\omega t} - \dot{b}\omega_0\right]$$

then substituting in for $\dot{a}(t)$. We have

$$\ddot{b} = \frac{i}{2}\left[(\frac{i}{2}(a\omega_0 + b\Omega e^{i\omega t}) - i\omega a)\Omega e^{-i\omega t} - \dot{b}\omega_0\right]$$

and

$$\ddot{b} = \frac{i}{2}\left[(ia\Omega e^{-i\omega t}(\frac{\omega_0}{2} - \omega) + i\frac{b}{2}\Omega^2 - \dot{b}\omega_0\right]$$

Then substitute the expression for $a(t)$ from the original equation

$$a(t)\Omega e^{-i\omega t} = -2i\dot{b}(t) + b(t)\omega_0.$$

and we get

$$\ddot{b} = \frac{i}{2}\left[(i\left((-2i\dot{b} + b\omega_0)e^{i\omega t}/\Omega\right)\Omega e^{-i\omega t}(\frac{\omega_0}{2} - \omega) + i\frac{b}{2}\Omega^2 - \dot{b}\omega_0\right]$$

$$\ddot{b} = \frac{i}{2}\left[(i\left(-2i\dot{b} + b\omega_0\ \right)(\frac{\omega_0}{2} - \omega) + i\frac{b}{2}\Omega^2 - \dot{b}\omega_0\right]$$

$$\ddot{b} = \frac{i}{2}\left[(-2\dot{b}\omega + i\frac{b}{2}\left(\omega_0^2 - 2\omega_0\omega + \Omega^2\right)\right]$$

$$\ddot{b} = -i\dot{b}\omega - \frac{b}{4}\left(\omega_0^2 - 2\omega_0\omega + \Omega^2\right)$$

$$\ddot{b} = -i\dot{b}\omega - \frac{b}{4}\left(\omega'^2 - \omega^2\right)$$

Define $\gamma = i\omega$, and $\alpha = \frac{b}{4}(\omega'^2 - \omega^2)$. Now our dcoupled second order differential equation looks like the equation of motion for a damped harmonic oscillator.

$$\ddot{b} = -\gamma\dot{b} - \alpha b$$

The general solution has the form

$$b = Ae^{i\theta t}$$

Substitution into the differential equation gives

$$-\theta^2 = -i\theta\gamma - \alpha$$

and

$$\theta = \frac{1}{2}\left(i\gamma \pm \sqrt{-\gamma^2 + 4\alpha}\right) = \frac{1}{2}\left(-\omega \pm \sqrt{\omega^2 + (\omega'^2 - \omega^2)}\right) = -\frac{\omega}{2} \pm \frac{\omega'^2}{2}$$

so

$$b = \left[Ce^{i\frac{\omega'}{2}t} + De^{-i\frac{\omega'}{2}t}\right]e^{-i\frac{\omega}{2}t}$$

or even better

$$b = \left[A\cos(\omega' t/2) + B\sin(\omega' t/2)\right]e^{-i\frac{\omega}{2}t}$$

The boundary conditions give us

$$b(0) = A = b_0$$

and

$$\begin{aligned}
\dot{b}(0) &= (-i\omega/2)A + (\omega'/2)B = \frac{i}{2}(\Omega a_0 - \omega_0 b_0) \\
&\rightarrow \omega b_0 + i\omega' B = (\omega_0 b_0 - \Omega a_0) \\
&\rightarrow B = \frac{i}{\omega'}\left[(\omega - \omega_0)b_0 + \Omega a_0\right]
\end{aligned}$$

Then

$$b = \left[b_0\cos(\omega' t/2) + \frac{i}{\omega'}\left[(\omega - \omega_0)b_0 + \Omega a_0\right]\sin(\omega' t/2)\right]e^{-i\frac{\omega}{2}t}$$

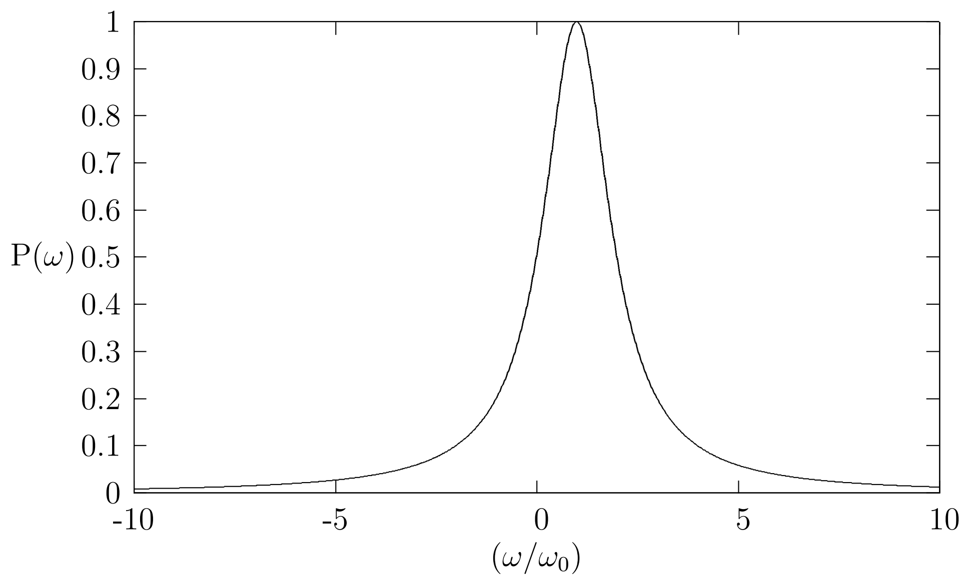

Figure 1: Transition probability as a function of driving frequency

(d) If $a_0 = 1$ and $b_0 = 0$ then

$$P(t) = |b(t)|^2 = |\frac{\Omega}{\omega'}|^2 \sin^2(\omega' t/2) = \frac{\omega^2}{(\omega-\omega_0)^2 + \Omega^2} \sin^2(\omega'/2)t$$

(e) The full width at half maximum $\Delta\omega = 2\Omega$.

(f)

$$\omega_0 = \gamma B_0 = \frac{ge}{2m_p} B_0 = \frac{(5.59)(1.6\times 10^{-19})\mathrm{C})}{2(1.67\times 10^{-27}\mathrm{kg})}(1T) = 2.68\times 10^8 \mathrm{Hz}$$

$$\Omega = \gamma B_{rf} = \frac{B_{rf}}{B_0}\omega_0 = 10^{-6}\omega_0 = 2.68\times 10^2 \mathrm{Hz}$$

5. **Griffiths 9.21**

(a) The interaction hamiltonian

$$\begin{aligned} H' &= -q\mathbf{E}(\mathbf{r},t)\cdot\mathbf{r} \\ &= -q\mathbf{E}_0(\cos\omega t + (\mathbf{k}\cdot\mathbf{r})\sin\omega t)\cdot\mathbf{r} \\ &= -q\hat{\epsilon}E_0(\cos\omega t + (\mathbf{k}\cdot\mathbf{r})\sin\omega t)\cdot\mathbf{r} \end{aligned}$$

The first term on the right gives us the dipole approximation and the corresponding spontaneous decay rate is

$$R = \frac{\omega^3 |q \langle \psi_f \mid \hat{\epsilon}\cdot\mathbf{r} \mid \psi_i\rangle|^2}{\pi\epsilon_0 \hbar c}$$

The second term on the right gives the magnetic dipole or electric quadrupole and the decay rate

$$\begin{aligned} R &= \frac{\omega^3 |q \langle \psi_f \mid (\hat{\epsilon}\cdot\mathbf{r})(\mathbf{k}\cdot\mathbf{r}) \mid \psi_i\rangle|^2}{\pi\epsilon_0\hbar c} \\ &= \frac{\omega^3 |q|\mathbf{k}| \left\langle \psi_f \mid (\hat{\epsilon}\cdot\mathbf{r})(\hat{\mathbf{k}}\cdot\mathbf{r}) \mid \psi_i\right\rangle|^2}{\pi\epsilon_0\hbar c} \\ &= \frac{\omega^5 |q \left\langle \psi_f \mid (\hat{\epsilon}\cdot\mathbf{r})(\hat{\mathbf{k}}\cdot\mathbf{r}) \mid \psi_i\right\rangle|^2}{\pi\epsilon_0\hbar c^3} \end{aligned}$$

where we use $|k| = \omega/c$

(b) For the 1-dimensional oscillator, we can choose $\mathbf{r}$ to be in the z-direction. The photon that is emitted can go in any direction $\mathbf{k}$ and with any polarization $\hat{\epsilon}$. Suppose that $\mathbf{k}$ is in the x-y plane at an angle θ with respect to $\mathbf{r}$. $\hat{k} \cdot \mathbf{r} = r\cos\theta$. The polarization is perpendicular to $\mathbf{k}$. If the polarization is in the x-y plane then $\hat{\epsilon} \cdot \mathbf{r} = r\sin\theta$. If it is at an angle ϕ with respect to the x-y plane then its projection into the plane is $\cos\phi$. So in general $\hat{\epsilon} \cdot \mathbf{r} = r\sin\theta\cos\phi$ and putting it all together

$$(\hat{\epsilon} \cdot \mathbf{r})(\hat{k} \cdot \mathbf{r}) = r^2\cos\theta\sin\theta\cos\phi$$

and

$$\begin{aligned} R &= \frac{q^2\omega^5}{\pi\epsilon\hbar c^5}|\left\langle f \mid r^2\cos\theta\sin\theta\cos\phi \mid i\right\rangle|^2 \\ &= \frac{q^2\omega^5}{\pi\epsilon\hbar c^5}|\left\langle f \mid r^2 \mid i\right\rangle|^2(\cos\theta\sin\theta\cos\phi)^2 \end{aligned}$$

The calculation of the expectation value does not depend on the angles since it is a 1-dimensional oscillator. And since it is 1-dimensional we might as let $r = x$. Then

$$R = \frac{q^2\omega^5}{\pi\epsilon\hbar c^5}|\left\langle f \mid x^2 \mid i\right\rangle|^2(\cos\theta\sin\theta\cos\phi)^2$$

$$\begin{aligned} \left\langle f \mid x^2 \mid i\right\rangle &= \left(\sqrt{\frac{\hbar}{2m\omega_0}}\right)^2 \left\langle f \mid (a_+ + a_-)^2 \mid i\right\rangle \\ &= \left(\frac{\hbar}{2m\omega_0}\right) \left\langle f \mid (a_+^2 - a_+a_- - a_-a_+ + a_-^2)^2 \mid i\right\rangle \end{aligned}$$

ω_0 is the natural frequency of the oscillator. Since the final state has less energy then the initial state, the only non-zero contribution comes from a_-^2. If the initial state is $\mid n\rangle$, then $a_-^2 \mid n\rangle = \sqrt{n(n-1)}\mid n-2\rangle$ and

$$\begin{aligned} R &= \frac{q^2\omega^5}{\pi\epsilon\hbar c^5}|\left(\frac{\hbar}{2m\omega_0}\right)\left\langle n-2 \mid a_-^2 \mid n\right\rangle|^2(\cos\theta\sin\theta\cos\phi)^2 \\ &= \frac{q^2\omega^5}{\pi\epsilon\hbar c^5}|\left(\frac{\hbar}{2m\omega_0}\right)\sqrt{n(n-1)}|^2(\cos\theta\sin\theta\cos\phi)^2 \\ &= \frac{q^2\omega^3\hbar}{\pi\epsilon m^2 c^5}n(n-1)(\cos\theta\sin\theta\cos\phi)^2 \end{aligned}$$

where we use the fact that $\omega_0 = \omega/2$. Finally, average over all angles for photon direction and polarization.

$$\frac{1}{4\pi} \int \int (\cos\theta \sin\theta \cos\phi)^2 \sin\theta d\theta d\phi = \frac{1}{15}$$

so

$$R = \frac{q^2 \omega^3 \hbar n(n-1)}{15\pi\epsilon m^2 c^5}$$

(c) To compute the rate for $2S \rightarrow 1S$ in hydrogen

$$\left\langle 1S \mid r^2 \cos\theta \sin\theta \cos\phi \mid 2S \right\rangle = \left\langle R(r)_{1S} \mid r^2 \mid R(r)_{2S} \right\rangle \int \int \cos\theta \sin\theta \cos\phi \sin\theta d\theta d\phi$$

The angular integral is zero. Note that the 1-dimensional harmonic oscillator has a preferred direction in space. The 1S and 2S states in hydrogen are spherically symmetric.